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II. *On the Equilibrium of Rotating Liquid Cylinders.*

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Introduction.

§ 1. As a preliminary to attacking the problem of determining the equilibrium configurations of a rotating mass of liquid, I was led to consider whether some method could not be devised for calculating the potential of a homogeneous mass in a manner more simple than that usually adopted. What was obviously required was a calculus enabling us to write down the potential of such a mass by an algebraical transformation of the equation of its boundary, instead of by an integration extending throughout its volume.

There was found to be no difficulty in reducing the calculation to a problem of algebraical transformation, but in three-dimensional problems the transformations required were, in general, as impracticable as the integrations which they were intended to replace. This was because the transformations depended upon a continued application of the formula which expresses the products or powers of spherical harmonics as the sum of a series of harmonics.

As soon, however, as we pass to the consideration of two-dimensional problems, the spherical harmonics may be replaced by circular functions of a single variable. The transformation now becomes manageable, and for this reason the present paper deals only with two-dimensional problems.

The first part of the paper contains a short sketch of a theory of two-dimensional potentials. I have, however, confined myself strictly to such problems as are required for the solution of the main problem under discussion, namely, that of the rotating liquid; the method does not attempt to be one of general applicability.

The Potentials of Homogeneous Cylinders.

General Theory.

§ 2. We shall suppose the cross-section of the cylinder of which the potential is required, to be bounded by a single continuous curve S enclosing the origin. Let

V be the potential of this cylinder, supposed to be composed of homogeneous matter of density ρ.

The value of V must be finite and continuous at all points except infinity, and its first differential coefficients must also be finite and continuous at all points. Also V must satisfy $\nabla^2 V = 0$ at all points outside S, and $\nabla^2 V = -4\pi\rho$ at all points inside S. At infinity V must vanish, except for a term proportional to $\log r$.

These conditions suffice to determine V uniquely. For if there were two distinct solutions V and V′, the function $(V - V')$ would satisfy $\nabla^2 (V - V') = 0$ at all points of space, would be finite and continuous, together with its first differential coefficients, at all points of space, and would vanish at infinity, except for a term proportional to $\log r$. The only solution satisfying these conditions is known to be $V - V' = 0$, hence any function V satisfying the conditions laid down above must be the potential of which we are in search.

§ 3. Let us use polar co-ordinates r, θ in conjunction with orthogonal co-ordinates x, y, and let us also introduce complex variables ξ, η, defined by

$$\xi = re^{i\theta} = x + iy, \qquad \eta = re^{-i\theta} = x - iy.$$

Let us suppose the equation to the curve S to be written in the form

$$f(\xi, \eta) = 0 \quad \ldots \ldots \quad (1),$$

and let us imagine that this equation is solved explicitly for ξ in the form

$$\xi = F(\eta) \quad \ldots \ldots \quad (2).$$

In general $F(\eta)$ will be a multiple-valued function of η, and the equation (1) may, and probably will, be satisfied for other values of ξ and η than those which occur on the surface S.

Let us, however, suppose that we have succeeded in finding one value of $F(\eta)$ such that this value is a single-valued function of η at every point of S, and is equal to ξ. Let us suppose that we have succeeded in expanding this value of $F(\eta)$ in a series of ascending and descending powers of η, these series each being supposed convergent at every point of S. Let us write

$$F(\eta) = \phi(\eta) + \psi(\eta) \quad \ldots \ldots \quad (3),$$

$$\phi(\eta) = a_0 + a_1\eta + a_2\eta^2 + \quad \ldots \quad (4), \qquad \psi(\eta) = \frac{b_1}{\eta} + \frac{b_2}{\eta^2} + \quad \ldots \quad (5).$$

We shall consider only the case in which the surface S has the plane $\theta = 0$ as a plane of symmetry. In this case the equation $f(\xi, \eta) = 0$ remains the equation to the curve after the sign of θ is changed, and we therefore have as a second form of this equation,

$$f(\eta, \xi) = 0 \quad \ldots \ldots \quad (6).$$

There is therefore a solution of this equation expressing η explicitly as a function of ξ in the form

$$\eta = F(\xi) = \phi(\xi) + \psi(\xi) \quad \ldots \ldots \quad (7)$$

where F, ϕ, ψ are defined by equations 3, 4 and 5.

Since we have supposed equation (2) to represent the complete solution of equation (1), the equation

$$\eta = \mathrm{F}(\xi) \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (8)$$

must necessarily represent the complete solution of equation (6), if the meaning of F remains unaltered. Let us suppose that the values of the multiple-valued function $\mathrm{F}(\xi)$ (or, what is the same thing, $\mathrm{F}(x+iy)$) are exhibited on the appropriate RIEMANN'S surface. The locus of points at which this function is equal to $x - iy$ (also a definite and unique function of position upon this RIEMANN'S surface) will be the complete system of points satisfying equation (6), and will therefore include the curve S. This curve will not, however, be drawn upon a plane, but upon a RIEMANN'S surface.

Now so long as the curve S does not possess a cusp or branch point we have at every point of S,

$$\left|\frac{d\mathrm{F}(\xi)}{d\xi}\right| = \left|\frac{d(x-iy)}{d(x+iy)}\right| = 1,$$

and hence it follows that no branch point of the RIEMANN'S surface can lie on the curve S. Let the curve S be a single closed curve surrounding the origin, and it follows that we can always so arrange the RIEMANN'S surface that no branch line shall intersect the curve S. In this case it is possible for a solution of the form of equation (6) to satisfy at every point of the curve S, the curve now being regarded as a closed curve lying on one sheet of a RIEMANN'S surface. By simply interchanging the axes of y and $-y$ we can imagine the function $\mathrm{F}(\eta)$ represented on the same RIEMANN'S surface, and we see that at every point on the old curve S we shall have $\xi = \mathrm{F}(\eta) = \phi(\eta) + \psi(\eta)$.

We have therefore proved that if a particular solution of the form of equation (3) can be found to represent the curve S, then solution (7) will represent the same curve. We have also seen that if a family of curves is described by continuous deformation starting from some curve S_0, the general solution can be arrived at by continuous variation starting from the solution for the curve S_0, *so long as no member of the family possesses a cusp or branch point.* For our present purpose it is sufficient to have proved that if we have, at every point of the surface S,

$$\xi = \phi(\eta) + \psi(\eta) \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (9),$$

then we have also, at every point,

$$\eta = \phi(\xi) + \psi(\xi) \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (10).$$

Let us now introduce a new function χ defined by

$$\chi = \mathrm{C} + \int_0^{\xi} \phi(\xi)\, d\xi - \int_{\xi}^{\infty} \psi(\xi)\, d\xi + \int_0^{\eta} \phi(\eta)\, d\eta - \int_{\eta}^{\infty} \psi(\eta)\, d\eta - \xi\eta \quad . \quad (11),$$

where C is a constant.

We have by differentiation,

$$\partial\chi/\partial\xi = \phi(\xi) + \psi(\xi) - \eta \quad . \quad . \quad (12), \qquad \partial\chi/\partial\eta = \phi(\eta) + \psi(\eta) - \xi \quad . \quad . \quad (13),$$

and therefore at the surface S, by equations (9) and (10), $\partial\chi/\partial\xi = \partial\chi/\partial\eta = 0$.

It follows that χ has a constant value at the surface S, and this, by a suitable choice of the constant C, may be taken to be zero. Also it follows that $\partial\chi/\partial n = 0$ at all points on the surface S, where $\partial/\partial n$ denotes differentiation with respect to the normal. We therefore have at every point of the surface S

$$\chi = 0 \quad \ldots \ldots \ldots \ldots (14), \qquad \partial\chi/\partial n = 0 \quad \ldots \ldots \ldots (15).$$

§ 4. Let us denote the potential at a point inside the surface S by V_i, and that at a point outside S by V_0. Let us examine, as a trial solution for V,

$$V_i = \pi\rho \left\{ C + \int_0^{\xi} \phi(\xi)\, d\xi + \int_0^{\eta} \phi(\eta)\, d\eta - \xi\eta \right\} \quad \ldots \ldots (16),$$

$$V_0 = \pi\rho \left\{ \int_{\xi}^{\infty} \psi(\xi)\, d\xi + \int_{\eta}^{\infty} \psi(\eta)\, d\eta \right\} \quad \ldots \ldots (17).$$

Since $\eta = re^{-i\theta}$, the greatest value of $|\eta|$ at any point on S is equal to the greatest radius, say R_1, which can be drawn from the origin to S. Since the series $\phi(\eta)$ is, by hypothesis, convergent for all points on the boundary, it follows that the radius of convergence of the power series $\phi(\eta)$ must be greater than R_1, and hence that $\phi(\eta)$ is convergent at all points inside the surface S. Hence also $\int_0^{\eta} \phi(\eta)\, d\eta$ must be convergent at all points inside S. The same is obviously true if ξ is written instead of η. Hence it follows that if V_i is defined by equation (16), then V_i and its first differential coefficients will be finite and continuous at all points inside S.

In a precisely similar manner it can be shown that if V_0 is defined by equation (17), then V_0 will be finite and continuous at all points outside S except at infinity, and that the first differential coefficients of V_0 will be finite at all points outside S.

From what has been said it follows that V_0 and V_i are finite at the boundary. The value of $V_i - V_0$ at the boundary is $\pi\rho\chi$, and this vanishes by equation (14). Hence V is finite and continuous at all points of space (except infinity).

Since the series $\phi(\eta)$ and $\psi(\eta)$ have been supposed to be convergent on the boundary, it follows that the first differential coefficients of V must be convergent on the boundary, and hence that these differential coefficients are finite at all points of space. At the boundary,

$$\partial V_i/\partial n - \partial V_0/\partial n = \pi\rho\, \partial\chi/\partial n = 0,$$

by equation (15). Hence it follows that the first differential coefficients of V are finite *and continuous* at all points of space.

At a point inside S,

$$\nabla^2 V = \nabla^2 V_i = 4\pi\rho \frac{\partial^2}{\partial\xi\, \partial\eta} \left\{ C + \int_0^{\xi} \phi(\xi)\, d\xi + \int_0^{\eta} \phi(\eta)\, d\eta - \xi\eta \right\} = -4\pi\rho,$$

and similarly at a point outside S,

$$\nabla^2 V = \nabla^2 V_0 = 4\pi\rho \frac{\partial^2}{\partial\xi\, \partial\eta} \left\{ \int_{\xi}^{\infty} \psi(\xi)\, d\xi + \int_{\eta}^{\infty} \psi(\eta)\, d\eta \right\} = 0.$$

Lastly, by actual integration, we find as the value of V_0 from equation (17),

$$V_0 = \pi\rho\left\{- b_1(\log\xi + \log\eta) + b_2\left(\frac{1}{\xi} + \frac{1}{\eta}\right) + \tfrac{1}{2}b_3\left(\frac{1}{\xi^2} + \frac{1}{\eta^2}\right) + \ldots\right\} \quad (18).$$

The term $- b_1(\log\xi + \log\eta)$ may be replaced by $- 2b_1\log r$; hence V vanishes at infinity, except for a term proportional to $\log r$.

We have now seen that V satisfies all the conditions which must be satisfied by a potential; hence the potential will be given by equations (16) and (17).

§ 5. Calculated by direct integration, the value of V_0 is

$$V_0 = -\rho\iint\log R^2 r'\,dr'\,d\theta',$$

where the integration extends throughout the cross-section of the cylinder, and R is the distance of the point r, θ at which the potential is being evaluated, from the point r', θ' of the cylinder. We have

$$R^2 = r^2 + r'^2 - 2rr'\cos(\theta - \theta') = (r - r'e^{i(\theta-\theta')})(r - r'e^{-i(\theta-\theta')}).$$

Now if $|r| > |r'|$, we have

$$\log(r - r'e^{-i(\theta-\theta')}) = \log r + \log\left(1 - \frac{r'e^{i\theta'}}{re^{i\theta}}\right) = \log r - \frac{r'e^{i\theta'}}{re^{i\theta}} - \frac{1}{2}\left(\frac{r'e^{i\theta'}}{re^{i\theta}}\right)^2 - \ldots,$$

and hence

$$\log R^2 = 2\log r - \left(\frac{r'e^{i\theta'}}{re^{i\theta}} + \frac{r'e^{-i\theta'}}{re^{-i\theta}}\right) - \ldots.$$

Upon integration we obtain

$$V_0 = -\rho\left\{2\log r\iint r'\,dr'\,d\theta' - \left(\frac{1}{\xi}\iint r'^2e^{i\theta'}dr'\,d\theta' + \frac{1}{\eta}\iint r'^2e^{-i\theta'}dr'\,d\theta'\right) - \ldots\right\} \quad (19).$$

The cylinder being symmetrical about the plane $\theta = 0$, we have

$$\iint r'^2e^{i\theta'}dr'\,d\theta' = \iint r'^2e^{-i\theta'}dr'\,d\theta' = \iint r'^2\cos\theta'\,dr'\,d\theta',$$

and hence equation (19) becomes

$$V_0 = -\rho(\log\xi + \log\eta)\iint r'\,dr'\,d\theta' + \rho\left(\frac{1}{\xi} + \frac{1}{\eta}\right)\iint r'^2\cos\theta'\,dr'\,d\theta' + \ldots \quad (20).$$

Let the area of the cross-section be A, and its centre of gravity be at $x = \alpha$, so that

$$A = \iint r'\,dr'\,d\theta' \qquad A\alpha = \iint r'^2\cos\theta'\,dr'\,d\theta;$$

then equation (15) becomes

$$V_0 = -\rho A(\log\xi + \log\eta) + \rho A\alpha\left(\frac{1}{\xi} + \frac{1}{\eta}\right) + \ldots.$$

Comparing this with equation (18), we find that

$A = \pi b_1$ . . . . (21), $\qquad A\alpha = \pi b_2$ . . . . (22).

These last two equations enable us to find, by a process of algebraical transformation, the cross-section and the centre of gravity of the cylinder whose boundary is given by equation (1). I have made no attempt to discuss these results from the point of view of pure mathematics; we therefore pass at once to some simple illustrations of our theory.

Circular Cylinder.

§ 6. Let us attempt to find the potential, cross-section, and centre of gravity of the cylinder, of which the cross-section is the circle

$$(x - c)^2 + y^2 = a^2 \quad \ldots \ldots \ldots \quad (23),$$

or, in polar co-ordinates,

$$r^2 = a^2 + 2cr\cos\theta - c^2 \quad \ldots \ldots \ldots \quad (24).$$

The ξ, η equation to this curve is

$$\xi\eta = a^2 + c(\xi + \eta) - c^2 \quad \ldots \ldots \ldots \quad (25),$$

or, solving explicitly for ξ,

$$\xi = c + a^2/(\eta - c).$$

Now the minimum value of $|\eta|$ which can occur on the curve is $a - c$, and this is greater than c if $a > 2c$. In this case we have

$$\frac{1}{\eta - c} = \frac{1}{\eta}\left(1 + \frac{c}{\eta} + \frac{c^2}{\eta^2} + \ldots\right),$$

and therefore the solution for ξ is

$$\xi = c + \frac{a^2}{\eta}\left(1 + \frac{c}{\eta} + \frac{c^2}{\eta^2} + \ldots\right) = c + \frac{a^2}{\eta} + \frac{ca^2}{\eta^2} + \ldots.$$

We accordingly have

$$V_i = \pi\rho\,[C + c(\xi + \eta) - \xi\eta] \quad \ldots \ldots \ldots \quad (26).$$

$$V_0 = \pi\rho\left\{- a^2 \log \xi\eta + ca^2\left(\frac{1}{\xi} + \frac{1}{\eta}\right) + \tfrac{1}{2}c^2a^2\left(\frac{1}{\xi^2} + \frac{1}{\eta^2}\right) + \ldots\right\} . \quad (27).$$

$$A = \pi a^2. \qquad A\alpha = \pi c a^2.$$

If we determine C from the condition that V_0 and V_i shall become equal at the boundary, we have the known values for V_0, V_i. The equations for A and α reduce to $A = \pi a^2$, $\alpha = c$, and these, again, are obviously in agreement with the known results.

Elliptic Cylinder.

§ 7. Let us next find the potential of the elliptic cylinder

$$ax^2 + by^2 = 1 \quad \ldots \ldots \ldots \quad (28).$$

The ξ, η equation to the surface is

$$(a - b)(\xi^2 + \eta^2) + 2(a + b)\xi\eta = 4,$$

or, solving explicitly for ξ,

$$\xi = \frac{1}{a - b}\left\{- (a + b)\eta + \sqrt{4ab\eta^2 + 4(a - b)}\right\} \quad \ldots \ldots \quad (29).$$

At points on the boundary the minimum value of $4ab\,|\eta^2|$ is $4a$; hence, provided $a^2 > (a-b)^2$, a convergent expansion for ξ is

$$\xi = \frac{1}{a-b}\left\{-(a+b)\,\eta + 2\sqrt{(ab)}\,\eta\left(1 + \tfrac{1}{2}\frac{a-b}{ab\eta^2} - \tfrac{1}{8}\frac{(a-b)^2}{a^2b^2\eta^4} - \ldots\right)\right\}$$

$$= \frac{\sqrt{a}-\sqrt{b}}{\sqrt{a}+\sqrt{b}}\,\eta + \frac{1}{\sqrt{(ab)}\,\eta} - \tfrac{1}{4}\frac{a-b}{(ab)^{3/2}\eta^3} - \ldots \qquad (30).$$

From this expansion we obtain at once

$$\mathrm{V}_i = -\frac{2\pi\rho}{\sqrt{a}+\sqrt{b}}\left(\sqrt{a}x^2 + \sqrt{b}y^2\right),$$

$$\mathrm{A} = \frac{\pi}{\sqrt{(ab)}}, \qquad \alpha = 0 \qquad (31).$$

We can obtain V_0 in series at once; if we require its value in finite terms we may proceed as follows:—

From equation (30),

$$\mathrm{V}_0 = \pi\rho\int_\eta^\infty\left(\frac{1}{\sqrt{(ab)}\,\eta} - \tfrac{1}{4}\frac{a-b}{(ab)^{3/2}\eta^3} + \ldots\right)d\eta + \text{same function of } \xi,$$

$$= \pi\rho\int_\eta^\infty\left[\frac{1}{a-b}\left\{-(a+b)\,\eta + \sqrt{4ab\eta^2 + 4(a-b)}\right\} + \frac{\sqrt{a}-\sqrt{b}}{\sqrt{a}+\sqrt{b}}\,\eta\right]d\eta$$
$$+ \text{same function of } \xi$$

$$= \frac{\pi\rho}{a-b}\left\{-\sqrt{(ab)}\,\eta^2 + \frac{\eta}{2}\sqrt{4ab\eta^2 + 4(a-b)}\right.$$
$$\left. + \frac{a-b}{\sqrt{(ab)}}\log\left(2\sqrt{(ab)}\,\eta + \sqrt{4ab\eta^2 + 4(a-b)}\right)\right\}$$
$$+ \text{same function of } \xi \quad . \quad . \quad . \qquad (32).$$

The results obtained agree with the known results if $\sqrt{a}$ and $\sqrt{b}$ are taken of the same sign. There is a second solution, obtained by changing the sign of one of these roots, and this corresponds to a mathematical ellipse of which one axis is negative. The full significance of this will appear later.

Expansion in Powers of a Parameter.

§ 8. When the equation to the surface is of a degree higher than the second, it will not, in general, be possible to obtain a solution in finite terms of the form of equation (29). Suppose, however, that the surface forms one of a family of surfaces, the family being described by the variation of a parameter c, and let this family be chosen so that the surface $c = 0$ is one for which the complete solution is known. Then as we know the value of ξ when $c = 0$, we shall assume that it will be possible to find the general value of ξ in a series of ascending powers of c, and the equations determining the various coefficients of c will have a unique solution.

The solution for ξ will be a function of η and c. The maximum values for $|\eta|$ and $\left|\frac{1}{\eta}\right|$ will therefore be functions of c, and will be expressible as series of ascending powers of c. Substituting these values respectively in $\phi(\eta)$ and $\psi(\eta)$, we obtain two series in ascending powers of c. Let the radii of convergence of these two series be c_1 and c_2 respectively. Then $\phi(\eta)$ is convergent inside and over the boundary provided $c < c_1$ and $\psi(\eta)$ is convergent outside and over the boundary provided $c < c_2$. The solution for ξ found by expansion in powers of c will satisfy the conditions which it is assumed to satisfy in § 3 so long as c is less than the smaller of the two quantities c_1 and c_2.

§ 9. Let us suppose our solution in powers of c to have been obtained. The values of V_i and V_0 can be regarded as power series of the variables ξ or η, and as such will have circles of convergence in space, having the origin as centre. Let V_i be convergent inside a circle of radius R_1, and V_0 outside a circle of radius R_2. When $c = 0$, $R_2 < R_1$; the circle R_2 being wholly inside, and the circle R_1 being wholly outside the surface S. When c reaches the value c_2, the circle R_2 touches the surface S; and for values of c greater than c_2, the circle R_2 either intersects, or lies outside, the surface S.

Suppose that $c_2 < c_1$: then for values of c such that

$$c_2 < c < c_1$$

the circle R_1 lies wholly outside S, while the circle R_2 does not lie wholly inside S. Thus if V_i and V_0 are defined by the same power series which have been found for them for values of c less than c_2, V_i will be convergent at all points inside S, but V_0 will not be convergent at all points outside S. We have now to inquire whether it is possible for V_i to give the true value of the internal potential, even when the series found for V_0 fails to represent the external potential.

Let the equation to the surface be written in the form

$$f(\xi, \eta, c) = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (33),$$

and let the solution be written in the form

$$\xi = \phi(\eta, c) + \psi(\eta, c) \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (34).$$

We shall only consider the case in which $c < c_1$, so that our former function $\phi(\eta)$ is, by hypothesis, convergent at the boundary and at all points inside it. We take $\phi(\eta, c)$ to mean the same thing as our former $\phi(\eta)$, and suppose $\psi(\eta, c)$ defined by equation (34). Now the value of ξ cannot become infinite at any point on the boundary, since we suppose the bounding surface not to extend to infinity for any value of c. Hence it follows that our function $\psi(\eta, c)$ will be finite at all points of the boundary so long as $c < c_1$. Moreover, in the region in which our former $\psi(\eta)$ is convergent (*i.e.*, in the region $r > R_2$), $\psi(\eta, c)$ must become identical with $\psi(\eta)$, and will therefore be finite. The function

$$\psi(\eta, c) + \psi(\xi, c)$$

will therefore be finite at the surface S, and will vanish at infinity to an order at least equal to $1/r$. This function is real at every point of space, and is a solution of LAPLACE'S equation. Hence the function is finite at every point of space.

It is therefore clear that if, in equations (16) and (17), we replace $\phi(\eta)$, $\psi(\eta)$ by $\phi(\eta, c)$, $\psi(\eta, c)$, and make the corresponding changes in $\phi(\xi)$, $\psi(\xi)$, we shall have a solution which satisfies all the conditions of the problem, subject to the single condition that c is less than c_1. At infinity $\psi(\eta, c)$ will be capable of expansion in the form of equation (5),

$$\psi(\eta, c) = b_1/\eta + b_2/\eta^2 + b_3/\eta^3 + \ldots,$$

and we now see that equations (21) and (22) can be obtained in the same manner as before.

It is therefore clear that the values obtained for V_i, A, and α will be the true values, even if they have been obtained by the use of divergent series, provided only that the series for V_i remains convergent up to the boundary.

In the case in which $c_1 < c_2$, a similar proposition is true for values of c such that $c_1 < c < c_2$.

§ 10. We now consider the case in which c is greater than either c_1 or c_2. Let the values of V_i and V_0 which have been found for values of c less than either c_1 or c_2 be extended, by a process of "continuation," to points outside their circles of convergence, and let the values so obtained define the functions V_i and V_0. These functions will have certain infinities, the position of these infinities depending upon the value of c. When $c = 0$, all the infinities of V_i lie outside S; all the infinities of V_0 lie inside S. Let us suppose that up to some value of c, say $c = c_3$, no infinity crosses S, but that (if possible) at $c = c_3$ one of these infinities is found on the boundary. The values of V_i and V_0 are functions of c, and $V_i - V_0$ satisfies the requisite algebraical equations from $c = 0$ until c is equal to the smaller of the values $c = c_1$ or c_2. Hence it must continue to satisfy for all values of c, until the condition found in § 3 is violated, *i.e.*, until the value of c is such that the curve possesses a cusp or branch point. Also V_i and V_0 satisfy the requisite conditions of finiteness, uniqueness, and continuity until c reaches the value c_3. Now as c approximates to c_3 from the direction in which $c < c_3$, the value of V at same point of the boundary (viz., the point at which the infinity occurs when $c = c_3$) will increase indefinitely, becoming ultimately infinite when $c = c_3$. This value of V will, however, give the true solution for all values of c less than c_3, and there will be a superior finite limit to the value of V. It therefore follows that there can be no value c_3 at which an infinity crosses the boundary, and the values of V_i and V_0 found by "continuation" of the power series will give the true values of V_i and V_0 until the whole solution is invalidated by the occurrence of a cusp or branch point.

Summing up, it appears that we may neglect the question of convergency of series altogether: so long as the values obtained for either V_i or V_0 are possible values, they must be true values. But care must be taken not to pass through values of the parameter such that S possesses a cusp or branch point for these values.

§ 11. Illustrations of these remarks are afforded by the examples of §§ 6 and 7. In equations (23), (24), and (25) let us regard c as a variable parameter, so that, as c varies, the equations represent the different members of a family of circular cylinders. We have seen that the series obtained for ξ only remains convergent so long as $c < \frac{1}{2}a$. Equations (26) and (27) represent the values of V_i and V_0 expanded in powers of c. Now the value of V_i is convergent, no matter how great c may be, hence we know that this represents the true value of V_i for all values of c. The value of V_0 has as its circle of convergence the circle $r = c$, and this intersects the boundary as soon as c attains the value $c = \frac{1}{2}a$. Hence the value obtained for V_0 will fail to give a true solution as soon as c exceeds the value $c = \frac{1}{2}a$, although it will always give the value of V_0 at points outside the circle $r = c$. Again, in § 7, let us regard $\sqrt{b}$ as a variable parameter. The value found for V_i is convergent for all values of $\sqrt{b}$, and therefore represents the true solution for all values of $\sqrt{b}$, provided that we start from a true solution, and do not pass through a value of $\sqrt{b}$ at which a cusp occurs. Under this same condition the series for V_0 will give the true value of V_0 at all points at which it is convergent, and the expression given in equation (32) will give the true value at all points, this being the expression for V_0 which would be found by "continuation" of the series, or (what is the same thing) by the methods of § 9.

The ellipse does not possess a cusp except in the critical cases of $\sqrt{b} = 0$, $\sqrt{b} = \infty$. In the former the ellipse reduces to a pair of parallel lines, and the points at infinity rank as cusps. In the latter the ellipse reduces to a doubled straight line joining the points $x = \pm a^{-\frac{1}{2}}$, and, again, these points rank as cusps. Hence a solution will remain the true solution so long as $\sqrt{b}$ does not pass through either of the values $\sqrt{b} = 0$ or ∞, *i.e.*, so long as $\sqrt{b}$ does not change sign. This is the meaning of the condition found in § 7, that $\sqrt{a}$ and $\sqrt{b}$ must be taken with the same sign.

We can see this from another point of view, as follows. The solution (29) can be exhibited on a RIEMANN's surface of two sheets, the branch points being given by

$$ab\eta^2 = b - a.$$

When $a = b$ (*i.e.*, when the ellipse reduces to a circle) these points coincide in the origin, and destroy one another. As b increases, the two points move along the axis of x, and ultimately meet the curve when $b = \infty$ at the points $x = \pm a^{-\frac{1}{2}}$, *i.e.*, meet the curve at its cusps as soon as cusps occur. Similarly, as b decreases from the value $b = a$, the branch points move along the axis of y, and meet the curve when $b = 0$ at the points $y = \pm\infty$.

Deformed Circular Cylinder.

§ 12. As an example of expansion in a series of powers of a parameter, let us consider the cylinder of which the equation is

$$r^2 = a^2 + 2cr^n \cos n\theta \quad \ldots \quad (35),$$

or, in ξ, η co-ordinates,

$$\xi\eta = a^2 + c(\xi^n + \eta^n) \quad \ldots \quad (36).$$

We are in search of a solution for ξ expanded in powers of c in the form

$$\xi = u_0 + u_1 c + u_2 c^2 + u_3 c^3 + \ldots \quad (37),$$

in which $u_0, u_1, \ldots$ are functions of η.

Substitute this assumed solution in equation (36), and we obtain

$$\begin{aligned}\eta(u_0 + u_1 c + u_2 c^2 + u_3 c^3 + \ldots) \\ = a^2 + c\{\eta^n + u_0{}^n + n u_0{}^{n-1} u_1 c + \tfrac{1}{2}n(n-1) u_0{}^{n-2} u_1{}^2 c^2 \\ + n u_0{}^{n-1} u_2 c^2 + \tfrac{1}{6} n(n-1)(n-2) u_0{}^{n-3} u_1{}^3 c^3 \\ + n(n-1) u_0{}^{n-2} u_1 u_2 c^3 + n u_0{}^{n-1} u_3 c^3\} \\ + \text{terms of degree 4 and higher in } c.\end{aligned}$$

Equating the coefficients of the various powers of c, we obtain

$$\eta u_0 = a^2, \qquad \eta u_1 = \eta^n + u_0{}^n,$$

$$\eta u_2 = n u_0{}^{n-1} u_1, \qquad \eta u_3 = \tfrac{1}{2} n(n-1) u_0{}^{n-2} u_1{}^2 + n u_0{}^{n-1} u_2,$$

$$\eta u_4 = \tfrac{1}{6} n(n-1)(n-2) u_0{}^{n-3} u_1{}^3 + n(n-1) u_0{}^{n-2} u_1 u_2 + n u_0{}^{n-1} u_3, \text{ \&c.}$$

Solving these equations in succession, we obtain

$$u_0 = \frac{a^2}{\eta}, \qquad u_1 = \eta^{n-1} + \frac{a^{2n}}{\eta^{n+1}}, \qquad u_2 = n a^{2n-2}\left(\frac{1}{\eta} + \frac{a^{2n}}{\eta^{n+1}}\right),$$

$$u_3 = \tfrac{1}{2} n(n-1) a^{2n-4} \eta^{n-1} + \frac{n(2n-1)}{\eta^{n+1}} a^{4n-4} + \frac{n(3n-1) a^{6n-4}}{2\eta^{3n+1}},$$

$$u_4 = \tfrac{1}{6} n(n-1)(n-2) a^{2n-6} \eta^{2n-1} + \frac{3n(n-1)}{2\eta} a^{4n-6},$$

$$+ \text{ terms of lower degree in } \eta, \text{ \&c.}$$

Hence we obtain at once

$$\begin{aligned}V_i = C + \pi\rho\Big\{-\xi\eta + \frac{\xi^n + \eta^n}{n} c + \tfrac{1}{2}(n-1) a^{2n-4}(\xi^n + \eta^n) c^3 \\ + \tfrac{1}{12}(n-1)(n-2) a^{2n-6}(\xi^{2n} + \eta^{2n}) c^4 + \ldots\Big\} \\ = C + \pi\rho\Big\{-r^2 + \left(\frac{2c}{n} + (n-1) a^{2n-4} c^3\right) r^n \cos n\theta \\ + \tfrac{1}{6}(n-1)(n-2) a^{2n-6} c^4 r^{2n} \cos 2n\theta + \ldots\Big\} \quad (38),\end{aligned}$$

$$A = \pi\{a^2 + n a^{2n-2} c^2 + \tfrac{3}{2} n(n-1) a^{4n-6} c^4 + \ldots\},$$

and, except in the special case of $n = 1$, $\alpha = 0$.

In this way we can, when c is small, write down the potential to any required degree of accuracy.

Deformed Elliptic Cylinder.

§ 13. As a final illustration, we shall find the potential produced by a small deformation of the surface of the elliptic cylinder

$$\xi\eta = a^2 + a_2(\xi^2 + \eta^2) \quad (39).$$

Let the deformed surface be

$$\xi\eta = a^2 + a_2(\xi^2 + \eta^2) + \sum_1^\infty b_n(\xi^n + \eta^n) \quad . \quad . \quad . \quad . \quad . \quad . \quad (40),$$

and let the solution be, as far as first powers of b's,

$$\xi = \xi_0 + \sum_1^\infty b_n \xi_n \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (41),$$

where ξ_n is a function of η, and ξ_0 is the solution when all the b's vanish.

Substitute solution (41) in equation (40), and equate coefficients of b_n, and we obtain $\xi_n\eta = 2a_2\,\xi_0\xi_n + \xi_0^n + \eta^n$, or, solving for ξ_n,

$$\xi_n = (\xi_0^n + \eta^n)/(\eta - 2a_2\,\xi_0) \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (42).$$

We can express ξ_0 in the form

$$\xi_0 = \alpha\eta + \beta/\eta + \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (43).$$

Hence equation (42) becomes

$$\xi_n = \frac{(1 + \alpha^n)\,\eta^n + n\alpha^{n-1}\,\beta\eta^{n-2} + \ldots}{\eta\,(1 - 2a_2\alpha) - 2a_2\,\beta\eta^{-1} + \ldots}$$

$$= \frac{1 + \alpha^n}{1 - 2a_2\alpha}\,\eta^{n-1} + \left(\frac{n\alpha^{n-1}\beta}{1 - 2a_2\alpha} + \frac{2a_2\beta\,(1 + \alpha^n)}{(1 - 2a_2\alpha)^2}\right)\eta^{n-3}, + \text{\&c.}$$

Hence we find as the value of V_i,

$$V_i = \pi\rho\,\{C - \xi\eta + \tfrac{1}{2}\alpha\,(\xi^2 + \eta^2)\}$$

$$+ \pi\rho\sum_{n=1}^{n=\infty}(\xi^n + \eta^n)\left\{\frac{1 + \alpha^n}{1 - 2a_2\alpha}\,\frac{b_n}{n} + \left(\frac{(n+2)\,\alpha^{n+1}\beta}{1 - 2a_2\alpha} + \frac{2a_2\beta\,(1 + \alpha^{n+2})}{(1 - 2a_2\alpha)^2}\right)\frac{b_{n+2}}{n}\right.$$

$$\left. + \text{ terms in } b_{n+4},\ b_{n+6},\ \text{\&c.} \ldots \right\} \quad . \quad . \quad . \quad . \quad (44).$$

The value of ξ_0 given by equation (43) is a solution of equation (39). Substituting this value, and equating the coefficients of the two highest powers of η, we find as the equations determining α and β,

$$\alpha = a_2\,(1 + \alpha^2), \qquad \beta = a^2 + a_2\,(2\alpha\beta + 1),$$

equations which will be required later.

Rotating Liquid Cylinder.

General Theory.

§ 14. We now pass to the main problem before us, and consider the equilibrium of a cylinder rotating with angular velocity ω.

The equation to the cylinder for a rotation equal to zero is

$$\xi\eta = a^2 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (45).$$

When the rotation ω is different from zero, we shall suppose the equation to the surface referred to its axis of rotation as origin to become

$$\xi\eta = a^2 + a_1(\xi + \eta) + a_2(\xi^2 + \eta^2) + \ldots \quad (46),$$

or, in polar co-ordinates,

$$r^2 = a^2 + 2a_1 r \cos\theta + 2a_2 r^2 \cos 2\theta + \ldots \quad (47).$$

This equation is not sufficiently general to represent all cylinders which are symmetrical about the initial line. The value of r^2 being known at the boundary, we shall always be able to find a function v such that v is finite and continuous, together with its first differential coefficients, at all points inside the boundary, and such that $\nabla^2 v = 0$ inside the boundary, and $v = r^2$ at the boundary. The value of v near the origin can be expanded in the form

$$v = a^2 + 2a_1 r \cos\theta + 2a_2 r^2 \cos 2\theta + \ldots \quad (48),$$

and this series will have a circle of convergence, say $r = \mathrm{R}$. It is only when the curve lies wholly inside this circle that the cylinder can be represented by an equation of the form of (47).

Let us, however, attach a conventional meaning to equation (47) in the case in which the right hand becomes divergent at the boundary, as follows. Let us suppose that the value of the function v given near the origin by equation (48) is calculated from its known values inside the circle $r = \mathrm{R}$, the values outside this circle being obtained by a process of "continuation." Then we shall suppose equation (47) to represent the locus of points at which $r^2 = v$.

Obviously, with this convention, equation (47) is sufficiently general to represent any surface. If this surface is to give an equilibrium configuration under a rotation ω, we must have

$$\mathrm{V}_i + \tfrac{1}{2}\omega^2 r^2 = \text{a constant} \quad (49)$$

at the surface. Now $\mathrm{V}_i + \pi\rho r^2$ is a spherical harmonic at all points inside the surface, and equation (49) can be written in the form

$$(\mathrm{V}_i + \pi\rho r^2) - \pi\rho\left(1 - \frac{\omega^2}{2\pi\rho}\right) r^2 = \text{a constant},$$

or, what is the same thing,

$$(\mathrm{V}_i + \pi\rho r^2) - \pi\rho\left(1 - \frac{\omega^2}{2\pi\rho}\right) v = \text{a constant} \quad (50).$$

This equation is satisfied at the surface S, and each term is a solution of LAPLACE's equation at every point inside S; hence the equation must be satisfied at every point inside S.

Now V_i can be calculated by the methods already explained, and we obtain an equation of the form

$$\mathrm{V}_i = \mathrm{C} - \pi\rho r^2 + \pi\rho \sum_{n=1}^{n=\infty} f_n(a_1, a_2, \ldots)(\xi^n + \eta^n),$$

which gives the value of V_i at all points inside a certain circle of convergence. The value of v inside its circle of convergence is, from equation (48),

$$v = a^2 + \sum_{n=1}^{n=\infty} a_n(\xi^n + \eta^n);$$

hence equation (50), at points inside both circles, becomes

$$C - \pi\rho a^2\left(1 - \frac{\omega^2}{2\pi\rho}\right)$$
$$+ \pi\rho \sum_{n=1}^{n=\infty} \left\{f_n(\alpha_1, \alpha_2, \ldots) - \alpha_n\left(1 - \frac{\omega^2}{2\pi\rho}\right)\right\}(\xi^n + \eta^n) = \text{a constant.} \quad (51).$$

We must therefore have, for all positive integral values of n,

$$f_n(\alpha_1, \alpha_2, \ldots) = \alpha_n\left(1 - \frac{\omega^2}{2\pi\rho}\right) \quad (52).$$

The condition that equation (46) or (47) may represent an equilibrium configuration is therefore

$$f_1/\alpha_1 = f_2/\alpha_2 = f_3/\alpha_3 = \ldots \quad (53),$$

and the value of ω^2 is such that each fraction is equal to $1 - \frac{\omega^2}{2\pi\rho}$. Since for any equilibrium configuration the axis of rotation must coincide with the centre of gravity, we see that if the coefficients of (46) satisfy (53), the curve must be referred to its centre of gravity as origin.

The points of bifurcation are given by the Hessian of this system of equations. For our purpose this may be most conveniently written in the form

$$\begin{vmatrix} \frac{\partial f_1}{\partial \alpha_1} - \left(1 - \frac{\omega^2}{2\pi\rho}\right), & \frac{\partial f_2}{\partial \alpha_1}, & \frac{\partial f_3}{\partial \alpha_1}, & \cdot & \cdot \\ \frac{\partial f_1}{\partial \alpha_2}, & \frac{\partial f_2}{\partial \alpha_2} - \left(1 - \frac{\omega^2}{2\pi\rho}\right), & \cdot & \cdot & \cdot \\ \frac{\partial f_1}{\partial \alpha_3}, & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot \end{vmatrix} = 0 \quad (54).$$

Our method will, for reasons already explained, break down as soon as a cusp or branch point occurs on any linear series. Any solution will be a true solution provided we can pass from this solution to another, known to be a true solution, over a path through a system of linear series, without passing through a point at which a cusp or branch point occurs. Now the occurrence of a cusp or branch point indicates, in the physical problem, the division of the mass of fluid into two separate masses, and when this occurs the solution breaks down for a second reason also: for equation (49) is only true when the surface of the fluid is continuous; when the surface consists of two distinct parts, the constant on the right-hand side has different values for the two parts of the surface.

This limitation will not cause trouble in the present investigation. For we are only desirous of tracing the changes in the configuration of the fluid up to the separation into two parts, and even if our method had enabled us to proceed beyond this point, it would have been fruitless to do so.

The Series of Circular Cylinders.

§ 15. If it were possible to calculate the f's and solve equations (53) in the most general case, we should arrive at a complete knowledge of the system of linear series of equilibrium configurations. This being impossible, we shall start from a known series, and calculate successive series by determining the various points of bifurcation.

Now we know (§ 12) that for the small values of the a's, f_n is of the form $f_n(a_1, a_2, \ldots) = \frac{a_n}{n}$. Hence there is a solution of the system of equations (52) given by $a_1 = a_2 = \ldots = 0$.

This is the series of circular configurations, and corresponds to the series of Maclaurin spheroids in the three-dimensional problem. When $\omega^2 > 2\pi\rho$ the solution breaks down physically, since the pressure at every point of the liquid becomes negative. In fact, when ω^2 reaches the value $\omega^2 = 2\pi\rho$ the series gives place to a series of annular forms, for each of which ω^2 has the critical value. We can adjust the radius of the annulus so as to give any desired amount of angular momentum greater than the critical value which occurs in the circular configuration when $\omega^2 = 2\pi\rho$.

Points of Bifurcation on Circular Series.

§ 16. To search for points of bifurcation on this linear series, we replace f_n by a_n/n in equation (54). Every term in the determinant on the left hand now vanishes, except the terms of the leading diagonal, and the equation reduces to

$$\left\{1 - \left(1 - \frac{\omega^2}{2\pi\rho}\right)\right\} \left\{\tfrac{1}{2} - \left(1 - \frac{\omega^2}{2\pi\rho}\right)\right\} \ldots \left\{\frac{1}{n} - \left(1 - \frac{\omega^2}{2\pi\rho}\right)\right\} \ldots = 0.$$

The different roots correspond to the different integral values of n, and are given by

$$n = 1, \quad 2, \quad 3, \quad 4, \quad 5, \ldots \infty,$$
$$\omega^2/2\pi\rho = 0, \quad \cdot 5, \quad \cdot 666, \quad \cdot 75, \quad \cdot 8, \ldots \; 1.$$

The first point of bifurcation ($n = 1$) may be rejected at once, the critical "vibration" being merely a displacement of the entire cylinder as a rigid body.

A displacement in which a_n only occurs for the single value $n = s$ will alter the potential energy only by a term proportional to the square of a_n. Hence the principal vibrations correspond to the different values of n from 2 to ∞, and are such that a_n only occurs for a single value of n in each. When $\omega^2 = 0$, all these vibrations are stable. When ω^2 reaches the point of bifurcation of order s, the vibration of order s becomes unstable, and, since there is only one point of bifurcation of order s, this vibration remains unstable for all values of ω^2 greater than the value at this point of bifurcation. We therefore see that by the time that ω^2 reaches the limiting value $2\pi\rho$, every vibration is unstable.

The Series of Elliptic Cylinders.

§ 17. The stable linear series is that of order 2. The initial deformation is therefore elliptical, and the point of bifurcation occurs for the value $\omega^2 = \pi\rho$.

The equation to the surface is initially

$$\xi\eta = a^2 + a_2(\xi^2 + \eta^2) \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (55).$$

In § 13 we found (equation (44)) as the corresponding value of V_i

$$V_i = \pi\rho\,\{C - \xi\eta + \tfrac{1}{2}\alpha\,(\xi^2 + \eta^2)\} \quad . \quad . \quad . \quad . \quad . \quad . \quad (56),$$

in which α is a root of

$$\alpha = a_2(1 + \alpha^2). \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (57),$$

and this is true however great a_2 may be. The values of $f_n(0, a_2, 0, 0, \ldots)$ can accordingly be written down at once. We have

$$f_2(0, a_2, 0, 0, \ldots) = \tfrac{1}{2}\alpha,$$

and all the other functions vanish.

Hence there is a general solution of equations (52) in which all the a_n's vanish except a_2, and

$$\tfrac{1}{2}\alpha = (1 - \omega^2/2\pi\rho)\,a_2 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (58),$$

where α is given by equation (57).

This is the linear series of which we are in search. It is obviously a series of elliptic cylinders, and corresponds to the series of Jacobian ellipsoids in the three-dimensional problem. From equations (57) and (58) we have

$$\omega^2/\pi\rho = 1 - \alpha^2. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (59).$$

We therefore see that as we move along this series the value of ω^2 continually decreases from $\pi\rho$ to 0. The angular momentum, however, increases from a finite to an infinite value.

The Remaining (Unstable) Series.

§ 18. Before searching for points of bifurcation on this series, let us briefly examine the series passing the other points of bifurcation on the circular series, these series being known to be all unstable. Near the point of bifurcation the form of the series of order n is

$$\xi\eta = a^2 + a_n(\xi^n + \eta^n) \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (60).$$

In § 11 we have calculated the values of $f_s(0, 0, \ldots a_n, 0, \ldots)$ as far as a_n^4. If we neglect a_n^4, it appears that all these functions vanish except f_n, and that f_n is given by

$$f_n(0, 0, \ldots a_n, 0, \ldots) = a_n/n + \tfrac{1}{2}(n-1)\,a^{2n-4}a_n^3.$$

The series is accordingly given by equation (60), until a_n^4 become appreciable.

The value of ω^2 is given by

$$1 - \omega^2/2\pi\rho = 1/n + \tfrac{1}{2}(n-1)\,a^{2n-4}\alpha_n^{\,2}.$$

It therefore appears that the angular velocity decreases as we recede from the point of bifurcation.

Throughout the series the bounding curve is periodic in θ with a period $2\pi/n$, and it is easily seen that at the remote end of the series is a curve consisting of n equal and symmetrically arranged arms, these arms extending in the limit to infinity, and being of zero breadth at all points except in the immediate neighbourhood of the origin.

In fig. 1 I have drawn the curve for the case of $n = 3$, $\alpha_3 = \dfrac{1}{4\sqrt{2a}}$. The error in the value of V_i (equation (38)) caused by the neglect of $\alpha_3^{\,4}$ will be seen to be of the order of $\cdot 0003\pi\rho\,\dfrac{r^6}{a^4}\cos 4\theta$.

Since the maximum value of r^6/a^4 on this curve is $8a^2$, it appears to be legitimate to neglect this error.

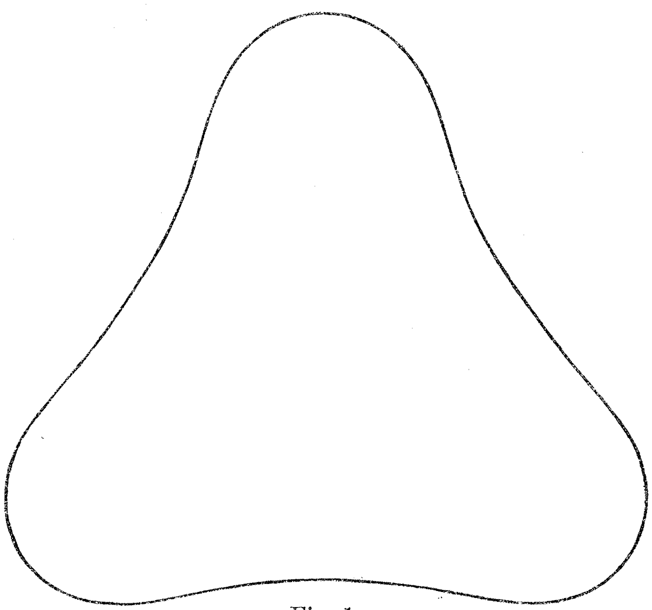

Fig. 1.

Points of Bifurcation on Elliptic Series.

§ 19. Let us now return to the series of elliptic cylinders, and search for points of bifurcation on this series. We have found in § 13 that when α_2 is finite, but when the squares and products of the remaining α's may be neglected, we have (equation (44)) for values of n different from 2,

$$f_n(\alpha_1, \alpha_2, \alpha_3, \ldots) = \frac{1 + a^n}{1 - 2a\alpha_2}\,\frac{\alpha_n}{n} + \text{terms linear in } \alpha_{n+2},\ \alpha_{n+4},\ \text{\&c.} \quad . \quad . \quad (61).$$

Let us now examine the form assumed by equation (54) at points upon this series. We have, in the first place, $\partial f_1/\partial\alpha_2 = 0$.

We have also $\partial f_m/\partial f_n = 0$ whenever m is greater than n. Hence we see that the determinant on the left hand of (54) reduces to the products of the terms in its leading diagonal, and that the equation itself is equivalent to the separate equations

$$\partial f_n/\partial a_n = (1 - \omega^2/2\pi\rho) \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (62)$$

taken for all values of n from 1 to ∞.

Corresponding to a root of (62) there is a point of bifurcation, and the linear series starting from the point must be found from equations (52). From these equations it appears that the linear series corresponding to a root of (62) will be such that, as far as the first order of small quantities, a_s exists only for the values $s = 2, n, n-2, n-4, \ldots$.

Of these series the series $n = 2$ may be rejected, as corresponding only to a step along the series of elliptic cylinders, and not to a new series at all, and the series $n = 1$ may be rejected, as corresponding merely to a change of origin.

We are left with the values $n = 3, 4, 5, \ldots$, and for any one of these values we have, from equation (61),

$$\frac{\partial f_n}{\partial a_n} = \frac{1}{n}\frac{1 + \alpha^n}{1 - 2\alpha a_2}.$$

The points of bifurcation are accordingly given by the equations

$$\frac{1}{n}\frac{1 + \alpha^n}{1 - 2\alpha a_2} = 1 - \frac{\omega^2}{2\pi\rho} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (63),$$

where n has the values $3, 4, 5, \ldots$.

These points of bifurcation are points on the series of elliptic cylinders, hence ω^2, a_2, and α are connected by equations (57) and (58). If we eliminate ω^2 and a_2 from the three equations (57), (58), and (63), we find, as the equation giving points of bifurcation of order n,

$$\frac{1 - \alpha^2}{2} = \frac{1 + \alpha^n}{n} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (64).$$

This equation must be solved by graphical methods. In fig. 2 the curve which is concave to the axis of α is the parabola

$$y = \tfrac{1}{2}(1 - \alpha^2) \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (65).$$

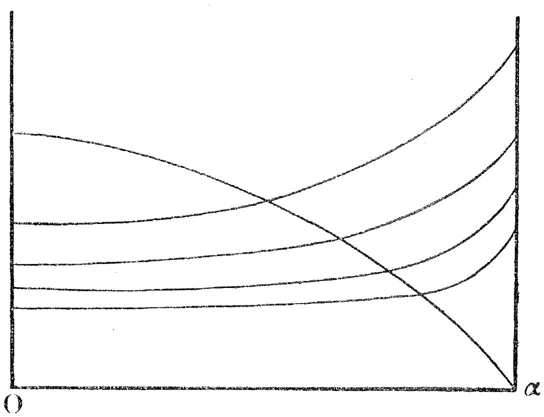

Fig. 2.

The remaining curves are the graphs of

$$y = \frac{1 + \alpha^n}{n} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (66)$$

for the values $n = 3, 4, 5, \ldots$.

The curve (66) cuts $\alpha = 0$ at the point $y = 1/n$, and $\alpha = 1$ at $y = 2/n$. The value of $dy/d\alpha$ is α^{n-1}. It is therefore obvious that the curves are convex to the axis of α, and since for any value of α the value of $dy/d\alpha$ is greatest for that curve for which n is least, it is obvious that the curves can never intersect.

We therefore see that the parabola (65) will meet each of the curves (66) once, and once only, for values of α between 0 and 1. Moreover the smaller n is, the smaller the value of α at the intersection.

As we move along the series of elliptic cylinders, the value of α increases from 0 to 1. Hence there will be an infinite number of points of bifurcation on this series, of orders 3, 4, 5, ... The point at which we arrive first is that of order $n = 3$; those of orders 4, 5, . . . follow in succession. As before, we find that the configuration at the end of the series of elliptic cylinders ($\alpha = 1$, an infinitely long and thin ellipse) is unstable for every vibration.

The linear series which we expect to be stable is that of order $n = 3$. To find the point of bifurcation of this series we require to solve the equation $\frac{1}{2}(1 - \alpha^2) = \frac{1}{3}(1 + \alpha^3)$ and the solution is found by inspection to be $\alpha = \frac{1}{2}$.

From equations (52) and (53) we find that at this point of bifurcation $\omega^2 = \frac{3}{4}\pi\rho$, and $a_2 = \frac{2}{5}$. The elliptic cylinder at the point of bifurcation is therefore the cylinder

$$\xi\eta = a^2 + \tfrac{2}{5}(\xi^2 + \eta^2). \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (67)$$

or, in Cartesian co-ordinates,

$$x^2 + 9y^2 = 5a^2 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (68).$$

If we reduce the linear scale of this until the area is a^2, we find for its equation

$$x^2 + 9y^2 = 3a^2,$$

and for its angular momentum, 1·46 times the greatest angular momentum for which the circular form is stable.

Poincaré's *Series of Pear-shaped Curves.*

§ 20. The configuration of the new linear series of order $n = 3$ is, near the point of bifurcation, of the form

$$\xi\eta = a^2 + \tfrac{2}{5}(\xi^2 + \eta^2) + b_3(\xi^3 + \eta^3) + b_1(\xi + \eta) \quad . \quad . \quad . \quad . \quad (69).$$

This new series is seen to be the series corresponding to Poincaré's series of pear-shaped figures.* Instead of making a separate problem out of the determination of the constants b_1 and b_3, we shall, in order to avoid repetition at a later stage, pass

* H. Poincaré, 'Acta Math.,' vol. 7, p. 347.

at once to the equations determining the general configuration of this series. We therefore replace equation (69) by

$$\xi\eta = a^2 + \tfrac{2}{5}(\xi^2 + \eta^2) + \sum_{s=1}^{s=\infty} \theta^s \left\{ {}_sC_0 + \sum_{n=1}^{n=\infty} {}_sC_n (\xi^n + \eta^n) \right\} \quad . \quad . \quad . \quad (70).$$

This will be assumed to be the general form of the surface in the linear series now under discussion, the quantity θ being a parameter which vanishes at the point of bifurcation.

The equation expressing explicitly the solution of (70) may be supposed to be

$$\xi = \left(1 - \frac{\omega^2}{2\pi\rho}\right)(\tfrac{8}{5}\xi_0 + \xi_1\theta + \xi_2\theta^2 + \xi_3\theta^3 + \ldots) \quad . \quad . \quad . \quad . \quad (71),$$

in which ξ_0 is the value of ξ when $\theta = 0$, and therefore satisfies

$$\xi_0\eta = a^2 + \tfrac{2}{5}(\xi_0^2 + \eta^2) \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (72),$$

and ξ_s is a series of ascending and descending powers of η, say

$$\xi_s = a_0 + a_1\eta + a_2\eta^2 + \ldots + a_{-1}\eta^{-1} + a_2\eta^{-2} + \ldots \quad . \quad . \quad . \quad (73).$$

If we calculate the value of V_i from (71), we find

$$V_i = -\pi\rho\xi\eta + \tfrac{8}{5}\pi\rho\left(1 - \frac{\omega^2}{2\pi\rho}\right)U_0 + \pi\rho\left(1 - \frac{\omega^2}{2\pi\rho}\right)\Sigma\theta^s U_s \quad . \quad . \quad (74),$$

where, if ξ_s is given by (73), the value of U_s is

$$U_s = C + a_0(\xi + \eta) + \tfrac{1}{2}a_1(\xi^2 + \eta^2) + \ldots \quad . \quad . \quad . \quad . \quad (75).$$

Using the value of V_i given by (74), the equation to be satisfied at the surface is

$$-\xi\eta(\pi\rho - \tfrac{1}{2}\omega^2) + \tfrac{8}{5}\pi\rho\left(1 - \frac{\omega^2}{2\pi\rho}\right)U_0 + \pi\rho\left(1 - \frac{\omega^2}{2\pi\rho}\right)\Sigma\theta^s U_s = \text{const.},$$

or, dividing throughout by $\pi\rho - \frac{1}{2}\omega^2$,

$$\xi\eta = \Sigma\theta^s U_s + \text{terms independent of } \theta \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (76).$$

Equation (76) must be identical with (70), the right-hand members of both being spherical harmonics, and hence we must have

$$U_s = {}_sC_0 + \sum_{n=1}^{n=\infty} {}_sC_n(\xi^n + \eta^n),$$

and therefore, by equation (75), ${}_sC_n = a_{n-1}/n$ for all positive values of n.

Instead of being given by equation (73), the value of ξ_s may now be supposed to be given by

$$\xi_s = {}_sC_1 + 2{}_sC_2\eta + 3{}_sC_3\eta^2 + \ldots + a_{-1}\eta^{-1} + a_{-2}\eta^{-2} + \ldots \quad . \quad . \quad (77).$$

If we introduce the limitation that the curve is to remain of constant area, we must put $a_{-1} = 0$. If we now replace a_{-2}, a_{-3} ... by new unknowns ${}_sC_{-1}$, ${}_sC_{-2}$, ..., we can write equation (77) in the symmetrical form

$$\xi_s = \sum_{n=-\infty}^{n=+\infty} n\,{}_sC_n\eta^{n-1} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (78),$$

in which we know that ${}_sC_{-1}$ must ultimately be equal to zero, in order that the centre of gravity may coincide with the origin (*cf.* equation 22).

Now ω^2 will be a function of θ, and the relation between ω^2 and θ is at our disposal, this relation being virtually the definition of θ. We have, however, already assumed that at the point of bifurcation $\theta = 0$ and $\partial\omega^2/\partial\theta = 0$. We therefore take as the relation between ω^2 and θ,

$$1 - \omega^2/2\pi\rho = \delta_0 + \delta_2\theta^2 + \delta_3\theta^3 + \ldots \quad (79),$$

in which $\delta_0 = \frac{5}{8}$, and δ_2, δ_3, ... are as yet undetermined.

The value of ξ given by equation (71) is now

$$\xi = \xi_0 + \theta\delta_0\xi_1 + \theta^2(\delta_0\xi_2 + \tfrac{8}{5}\delta_2\xi_0) + \theta^3(\delta_0\xi_3 + \delta_2\xi_1 + \tfrac{8}{5}\delta_3\xi_0) + \ldots \quad (80).$$

If we substitute this value in equation (70), of which we are supposing it to be a solution, we obtain

$$\begin{aligned}\eta\,(\xi_0 + \theta\delta_0\xi_1 + \theta^2(\delta_0\xi_2 + \tfrac{8}{5}\delta_2\xi_0) + \ldots)\\ = a^2 + \tfrac{2}{5}\{\eta^2 + \xi_0{}^2 + 2\xi_0(\theta\delta_0\xi_1 + \theta^2(\delta_0\xi_2 + \tfrac{8}{5}\delta_2\xi_0) + \ldots)\\ + (\theta\delta_0\xi_1 + \theta^2(\delta_0\xi_2 + \tfrac{8}{5}\delta_2\xi_0) + \ldots)^2\}\\ + \sum_{s=1}^{s=\infty}\theta^s\left({}_s\mathrm{C}_0 + \sum_{n=1}^{n=\infty}{}_s\mathrm{C}_n(\xi^n + \eta^n)\right) \quad (81),\end{aligned}$$

and if we equate the coefficients of successive powers of θ in this, we obtain

$$\xi_0\eta = a^2 + \tfrac{2}{5}(\xi_0{}^2 + \eta^2) \quad (82),$$

$$\delta_0\xi_1(\eta - \tfrac{4}{5}\xi_0) = {}_1\mathrm{C}_0 + \sum_1^\infty {}_1\mathrm{C}_n(\xi_0{}^n + \eta^n) \quad (83),$$

$$\begin{aligned}(\delta_0\xi_2 + \tfrac{8}{5}\delta_2\xi_0)(\eta - \tfrac{4}{5}\xi_0) = \tfrac{2}{5}(\delta_0\xi_1)^2 + \delta_0\xi_1\sum_1^\infty n\,{}_1\mathrm{C}_n\,\xi_0{}^{n-1}\\ + {}_2\mathrm{C}_0 + \sum_1^\infty {}_2\mathrm{C}_n(\xi_0{}^n + \eta^n) \quad (84),\end{aligned}$$

$$\begin{aligned}(\delta_0\xi_3 + \delta_2\xi_1 + \tfrac{8}{5}\delta_3\xi_0)(\eta - \tfrac{4}{5}\xi_0) = \tfrac{4}{5}\delta_0\xi_1(\delta_0\xi_2 + \tfrac{8}{5}\delta_2\xi_0)\\ + (\delta_0\xi_1)^2\sum_1^\infty \tfrac{1}{2}n(n-1)\,{}_1\mathrm{C}_n\,\xi_0{}^{n-2} + (\delta_0\xi_2 + \tfrac{8}{5}\delta_2\xi_0)\sum_1^\infty n\,{}_1\mathrm{C}_n\,\xi_0{}^{n-1}\\ + \delta_0\xi_1\sum_1^\infty n\,{}_2\mathrm{C}_n\,\xi_0{}^{n-1} + {}_3\mathrm{C}_0 + \sum_1^\infty {}_3\mathrm{C}_n(\xi_0{}^n + \eta^n). \quad (85).\end{aligned}$$

Equation (82) is, as it ought to be, identical with (72). Equation (83) enables us to determine the constants which occur in ξ_1. These having been found, equation (84) enables us to determine ξ_2, and so on in succession.

§ 21. As far as first powers of θ, we know that the configuration is of the form given by equation (69). We therefore assume at once for ξ_1 the form

$$\xi_1 = 3c_3\eta^2 + c_1 - \frac{c_{-1}}{\eta^2} - \frac{3c_{-3}}{\eta^4} - \ldots \quad (86),$$

in which c_n is temporarily written for ${}_1\mathrm{C}_n$.

Since $\delta_0 = \frac{5}{8}$, and since the c's higher then c_3 must vanish, we find that equation (83) takes the form

$$\tfrac{5}{8}(\eta - \tfrac{4}{5}\xi_0)\,\xi_1 = c_3(\xi_0{}^3 + \eta^3) + c_1(\xi_0 + \eta) \quad (87).$$

Since ξ_0 satisfies equation (82), we have

$$\xi_0 = \tfrac{5}{4}\eta \pm \tfrac{1}{2}\sqrt{(\tfrac{9}{4}\eta^2 - 10a^2)} \quad . \quad . \quad . \quad (88),$$

or, expanding in the appropriate form, and taking $a = 1$,

$$\xi_0 = \tfrac{1}{2}\eta + \frac{5}{3\eta} + \frac{50}{27\eta^3} + \frac{1000}{3^5\eta^5} + \frac{25{,}000}{3^7\eta^7} \quad . \quad . \quad . \quad (89).$$

From equations (82) and (89) we have

$$\xi_0^{\,2} = \tfrac{5}{2}\xi_0\eta - \eta^2 - \tfrac{5}{2} = \tfrac{1}{4}\eta^2 + \tfrac{5}{3} + \frac{125}{27\eta^2} + \frac{2500}{3^5\eta^4} + \ldots \quad . \quad . \quad (90).$$

By a similar process we obtain

$$\xi_0^{\,3} = (\tfrac{21}{4}\eta^2 - \tfrac{5}{2})\,\xi_0 - \tfrac{5}{2}\eta^3 - \tfrac{25}{4}\eta = \tfrac{1}{8}\eta^3 + \tfrac{5}{4}\eta + \frac{50}{9\eta} + \frac{1375}{81\eta^3} + \ldots \quad . \quad (91).$$

Equation (87) can now be put into the form

$$\left[\tfrac{3}{8}\eta - \frac{5}{6\eta} - \frac{25}{27\eta^3} - \frac{500}{243\eta^5} - \ldots\right]\left[3c_3\eta^2 + c_1 - \frac{c_{-1}}{\eta_2} - \frac{3c_{-3}}{\eta^4} - \ldots\right]$$

$$= c_3\left[\tfrac{9}{8}\eta^3 + \tfrac{5}{4}\eta + \frac{50}{9\eta} + \frac{1375}{81\eta^3} + \ldots\right]$$

$$+ c_1\left[\tfrac{3}{2}\eta + \frac{5}{3\eta} + \frac{50}{27\eta^3} + \ldots\right] \quad . \quad . \quad . \quad (92).$$

Equating the coefficients of the various powers of η we obtain

$$\tfrac{9}{8}c_3 = \tfrac{9}{8}c_3$$

$$-\tfrac{5}{2}c_3 + \tfrac{3}{8}c_1 = \tfrac{5}{4}c_3 + \tfrac{3}{2}c_1$$

$$-\tfrac{25}{9}c_3 - \tfrac{5}{6}c_1 - \tfrac{3}{8}c_{-1} = \tfrac{50}{9}c_3 + \tfrac{5}{3}c_1$$

$$-\tfrac{500}{81}c_3 - \tfrac{25}{27}c_1 + \tfrac{5}{6}c_{-1} - \tfrac{9}{8}c_{-3} = \tfrac{1375}{81}c_3 + \tfrac{50}{27}c_1.$$

The first equation is, as it ought to be, an identity. We may assign to c_3 any value, and therefore take $c_3 = 1$, this being equivalent to fixing the linear scale of measurement of θ. Solving the remaining equations in succession, we obtain the following scheme of values :—

$$c_3 = 1, \qquad c_1 = -\tfrac{10}{3}, \qquad c_{-1} = 0, \qquad c_{-3} = -\tfrac{1000}{81}.$$

The vanishing of c_{-1} shows that the centre of gravity of the curve is, as it ought to be, at the origin. We now have as the value of ξ_1, equation (86),

$$\xi_1 = 3\eta^2 - \tfrac{10}{3} + \frac{1000}{27\eta^4} + \ldots \quad . \quad . \quad . \quad (93).$$

§ 22. We now proceed to the determination of ξ_2. Equation (84) takes the form

$$(\eta - \tfrac{4}{5}\xi_0)\,(\delta_0\xi_2 + \tfrac{8}{5}\delta_2\xi_0) = \tfrac{2}{5}\delta_0^{\,2}\xi_1^{\,2} + \delta_0\xi_1\,(3\xi_0^{\,2} - \tfrac{10}{3})$$

$$+ {}_2C_0 + \sum_{n=1}^{n=\infty} {}_2C_n\,(\xi_0^{\,n} + \eta^n) \quad . \quad . \quad . \quad (94).$$

The value of ξ_2 is of the form (equation (78)),

$$\xi_2 = \sum_{-\infty}^{+\infty} n_2 C_n \eta^{n-1} \quad . \quad . \quad . \quad (95).$$

Suppose this value substituted for ξ_2 in equation (94), and in the equation so obtained equate the coefficients of the various powers of η. For any power of η greater than the fourth it will be seen that equation (94) may be replaced by

$$\delta_0 (\eta - \tfrac{4}{5}\xi_0) \sum_{n=-\infty}^{n=+\infty} n_2 C_n \eta^{n-1} - \sum_{n=1}^{n=\infty} {}_2C_n (\xi_0^n + \eta^n) = 0.$$

The equation obtained by equating to zero the coefficient of any power of η greater than the fourth will therefore be of the form

a linear homogeneous function of ${}_2C_n$, ${}_2C_{n+1}$, . . . = 0,

and there is an equation of this form for every value of n greater than 4. This system of equations can only be satisfied by taking

$${}_2C_5 = {}_2C_6 = {}_2C_7 = \ldots = 0.$$

We may therefore assume for ξ_2 (equation (95)) an expansion of the form

$$\xi_2 = 4d_4\eta^3 + 3d_3\eta^2 + 2d_2\eta + d_1 - \frac{d_{-1}}{\eta^2} - \frac{2d_{-2}}{\eta^3} \quad . \quad . \quad . \quad . \quad (96),$$

in which d_n is written for ${}_2C_n$.

From equation (82) we have

$$\delta_0\xi_1 (3\xi_0^2 - \tfrac{10}{3}) = \xi_1 (\tfrac{75}{16}\xi_0\eta - \tfrac{15}{8}\eta^2 - \tfrac{325}{48}),$$

and from equation (87),

$$(\eta - \tfrac{4}{5}\xi_0)\, \xi_1 = \tfrac{8}{5} (\xi_0^3 + \eta^3) - \tfrac{16}{3} (\xi_0 + \eta) \quad . \quad . \quad . \quad . \quad . \quad . \quad (97).$$

From these last two equations we obtain

$$\delta_0\xi_1 (3\xi_0^2 - \tfrac{10}{3}) = \xi_1 (\tfrac{255}{64}\eta^2 - \tfrac{325}{48}) - \tfrac{75}{8}\eta (\xi_0^3 + \eta^3) + \tfrac{125}{4}\eta (\xi_0 + \eta) \quad . \quad . \quad (98)$$

With the help of equations (97) and (98) we can write equation (94) in the form

$$\begin{aligned}(\eta - \tfrac{4}{5}\xi_0) (\delta_0\xi_2 + \tfrac{8}{5}\delta_2\xi_0) &= \tfrac{5}{32}\xi_1^2 + \xi_1 (\tfrac{255}{64}\eta^2 - \tfrac{325}{48}) \\ &\quad - \tfrac{75}{8}\eta (\xi_0^3 + \eta^3) + \tfrac{125}{4}\eta (\xi_0 + \eta) \\ &\quad + d_4 (\xi_0^4 + \eta^4) + d_3 (\xi_0^3 + \eta^3) + d_2 (\xi_0^2 + \eta^2) + d_1 (\xi_0 + \eta) + d_0 \quad . \quad (99).\end{aligned}$$

It is clear upon examination of this equation that the equations found upon equating the coefficients of η^3, η, η^{-1}, &c., will contain only terms multiplied by d_3, d_0, d_{-1}, &c., without constant terms. We therefore take

$$d_3 = d_1 = d_{-1} = \ldots = 0.$$

Equation (99) now contains only even powers of η. Before we can calculate the coefficients of these powers we must obtain series for ξ_1^2 and ξ_0^4. By squaring equation (93) we get

$$\xi_1^2 = 9\eta^4 - 20\eta^2 + \tfrac{100}{9} + \frac{2000}{9\eta^2} + \ldots \quad . \quad . \quad . \quad . \quad . \quad . \quad (100).$$

Next we have from equation (82)

$$\xi_0^2 + \eta^2 = \tfrac{5}{2} (\xi\eta - 1).$$

Squaring this, and subtracting $2\xi_0^2\eta^2$ from each side,

$$\xi_0{}^4 + \eta^4 = \tfrac{25}{4} - \tfrac{25}{2}\xi_0\eta + \tfrac{17}{4}\xi_0{}^2\eta^2.$$

Using the series which have already been obtained for $\xi_0{}^2$ and ξ_0 (equations (89) and (90)), we obtain

$$\xi_0{}^4 = \tfrac{1}{16}\eta^4 + \tfrac{5}{6}\eta^2 + \tfrac{275}{54} + \frac{5000}{243\eta^2} + \dots \ .$$

We can now evaluate that part of the right-hand side of equation (99) which does not contain d_4, d_2, or d_0. We have

$$\tfrac{5}{32}\xi_1{}^2 = \quad \tfrac{45}{32}\eta^4 - \tfrac{25}{8}\eta^2 + \tfrac{125}{72} + \frac{625}{18\eta^2} + \dots \ ,$$

$$\xi_1\left(\tfrac{255}{64}\eta^2 - \tfrac{325}{48}\right) = \quad \tfrac{765}{64}\eta^4 - \tfrac{1125}{32}\eta^2 + \tfrac{1625}{72} + \frac{10,625}{72\eta^2} + \dots \ ,$$

$$-\tfrac{75}{8}\eta\left(\xi_0{}^3 + \eta^3\right) = -\tfrac{675}{64}\eta^4 - \tfrac{375}{32}\eta^2 - \tfrac{625}{12} - \frac{34,375}{216\eta^2} + \dots \ ,$$

$$\tfrac{125}{4}\eta\left(\xi_0 + \eta\right) = \quad \tfrac{375}{8}\eta^2 + \tfrac{625}{12} + \frac{3125}{54\eta^2} + \dots \ .$$

By addition the sum of the terms in question is found to be

$$\tfrac{45}{16}\eta^4 - \tfrac{25}{16}\eta^2 + \tfrac{875}{36} + \frac{4375}{54\eta^2} + \dots \ .$$

Lastly, we have

$$\tfrac{8}{5}\left(\delta_0\xi_2 + \tfrac{8}{5}\delta_2\xi_0\right) = 4d_4\eta^3 + \left(2d_2 + \tfrac{32}{25}\delta_2\right)\eta + \tfrac{64}{15}\delta_2\eta^{-1} + \left(-2d_{-2} + \tfrac{128}{27}\delta_2\right)\eta^{-3} + \dots$$

Collecting the various series, we find as the form assumed by equation (99),

$$\left(\tfrac{3}{8}\eta - \frac{5}{6\eta} - \frac{25}{27\eta^3} - \frac{500}{3^5\eta^5} - \dots\right)$$

$$\left(4d_4\eta^3 + \left(2d_2 + \tfrac{32}{25}\delta_2\right)\eta + \tfrac{64}{15}\delta_2 + \left(-2d_{-2} + \tfrac{128}{27}\delta_2\right)\eta^{-3} + \dots\right)$$

$$= \tfrac{45}{16}\eta^4 - \tfrac{25}{16}\eta^2 + \tfrac{875}{36} + \frac{4375}{54\eta^2} + \dots$$

$$+ d_4\left(\tfrac{17}{16}\eta^4 + \tfrac{5}{6}\eta^2 + \tfrac{275}{54} + \frac{5000}{243\eta^2} + \dots\right)$$

$$+ d_2\left(\tfrac{5}{4}\eta^2 + \tfrac{5}{3} + \frac{125}{27\eta^2} + \dots\right)$$

$$+ d_0 .$$

Equating the coefficients of the various powers of η, we obtain

$$\tfrac{3}{4}d_4 = \quad \tfrac{45}{16} + \tfrac{17}{16}d_4 ,$$

$$-\tfrac{10}{3}d_4 + \tfrac{3}{4}d_2 + \tfrac{12}{25}\delta_2 = -\tfrac{25}{16} + \tfrac{5}{6}d_4 + \tfrac{5}{4}d_2 ,$$

$$-\tfrac{100}{27}d_4 - \tfrac{5}{3}d_2 + \tfrac{8}{15}\delta_2 = \quad \tfrac{875}{36} + \tfrac{275}{54}d_4 + \tfrac{5}{3}d_2 + d_0 ,$$

$$-\tfrac{2000}{243}d_4 - \tfrac{50}{27}d_2 - \tfrac{80}{27}\delta_2 - \tfrac{3}{4}d_{-2} = \tfrac{4375}{54} + \tfrac{5000}{243}d_4 + \tfrac{125}{27}d_2 .$$

Solving, we obtain in succession,

$$d_4 = \tfrac{45}{7}, \qquad d_2 = -\tfrac{2825}{56} + \tfrac{24}{25}\delta_2.$$
$$d_0 = \tfrac{5500}{63} - \tfrac{8}{3}\delta_2, \qquad d_{-2} = \tfrac{4375}{54} - \tfrac{992}{81}\delta_2.$$

We therefore find as the value of ξ_2

$$\xi_2 = \tfrac{180}{7}\eta^3 - \tfrac{2825}{28}\eta - \frac{4375}{27\eta^3} + \ldots + \delta_2\left\{\tfrac{48}{25}\eta + \frac{1984}{81\eta^3} + \ldots\right\} \quad . \quad (101).$$

§ 23. We now proceed to the determination of ξ_3. Equation (85) takes the form

$$(\delta_0\xi_3 + \delta_2\xi_1 + \tfrac{8}{5}\delta_3\xi_0)(\eta - \tfrac{4}{5}\xi_0)$$
$$= \tfrac{4}{5}\delta_0\xi_1(\delta_0\xi_2 + \tfrac{8}{5}\delta_2\xi_0)$$
$$+ (\delta_0\xi_1)^2 3\xi_0 + (\delta_0\xi_2 + \tfrac{8}{5}\delta_2\xi_0)(3\xi_0^2 - \tfrac{10}{3})$$
$$+ \delta_0\xi_1(\tfrac{180}{7}\xi_0^3 - \tfrac{2825}{28}\xi_0) + \delta_0\delta_2\tfrac{48}{25}\xi_0\xi_1$$
$$+ {}_3C_0 + \sum_1^\infty {}_3C_n(\xi_0^n + \eta^n) \quad . \quad . \quad . \quad (102).$$

All the terms on the right-hand side, except those in the last line, are of odd degree in η and of degree 5 at most. The same is true of the terms on the left-hand which are multiplied by δ_2. The terms multiplied by δ_3 are of even degree, two at most. It is therefore clear that we may at once take

$${}_3C_6 = {}_3C_7 = {}_3C_8 = {}_3C_9 = \ldots = 0,$$
$$\delta_3 = 0\,; \quad {}_3C_4 = {}_3C_2 = {}_3C_0 = \ldots = 0,$$

and assume for ξ_3 an expansion of the form

$$\xi_3 = 5e_5\eta^4 + 3e_3\eta^2 + e_1 - \frac{e_{-1}}{\eta^2} - \ldots \quad . \quad . \quad . \quad (103).$$

We now calculate the various series which occur in equation (102). We have

$$\delta_0(\eta - \tfrac{4}{5}\xi_0)\xi_3 = \left[\tfrac{3}{8}\eta - \frac{5}{6\eta} - \frac{25}{27\eta^3} - \frac{500}{243\eta^5} - \ldots\right]\xi_3,$$

and from equation (97),

$$\delta_2(\eta - \tfrac{4}{5}\xi_0)\xi_1 = \delta_2[\tfrac{8}{5}(\xi_0^3 + \eta^3) - \tfrac{16}{3}(\xi_0 + \eta)]$$
$$= -\delta_2\frac{64}{375\eta}[-\tfrac{75}{8}\eta(\xi_0^3 + \eta^3) + \tfrac{125}{4}\eta(\xi_0 + \eta)].$$

This last bracket can be at once calculated from the series of the last page; we have

$$\delta_2(\eta - \tfrac{4}{5}\xi_0)\xi_1 = \delta_2(\tfrac{9}{5}\eta^3 - 6\eta + 0\,.\,\eta^{-1} + \ldots) \quad . \quad . \quad . \quad (104).$$

Next

$$\tfrac{4}{5}\delta_0\xi_1(\delta_0\xi_2 + \tfrac{8}{5}\delta_2\xi_0) + (\delta_0\xi_2 + \tfrac{8}{5}\delta_2\xi_0)(3\xi_0^2 - \tfrac{10}{3}) = (\tfrac{5}{8}\xi_2 + \tfrac{8}{5}\delta_2\xi_0)(\tfrac{1}{2}\xi_1 + 3\xi_0^2 - \tfrac{10}{3})$$

From equations (93) and (90) we have

$$\tfrac{1}{2}\xi_1 + 3\xi_0^2 - \tfrac{10}{3} = \left(\tfrac{3}{2}\eta^2 - \tfrac{5}{3} + \frac{500}{27\eta^4}\right) + \left(\tfrac{3}{4}\eta^2 + 5 + \frac{125}{9\eta^2} + \frac{2500}{81\eta^4}\right) - \tfrac{10}{3}$$
$$= \tfrac{9}{4}\eta^2 + \frac{125}{9\eta^2} + \frac{4000}{81\eta^4} \quad . \quad . \quad . \quad (105).$$

From equations (101) and (89) we have

$$\tfrac{5}{8}\xi_2 + \tfrac{8}{5}\delta_2\xi_0 = \tfrac{5}{8}\left\{\tfrac{180}{7}\eta^3 - \tfrac{2825}{28}\eta - \frac{4375}{27\eta^3} - \ldots\right\} + \tfrac{5}{8}\delta_2\left\{\tfrac{48}{25}\eta + \frac{1984}{81\eta^3} + \ldots\right\}$$

$$+ \delta_2\left\{\tfrac{4}{5}\eta + \frac{8}{3\eta} + \frac{80}{27\eta^3} + \ldots\right\}$$

$$= \tfrac{225}{14}\eta^3 - \tfrac{14,125}{224}\eta - \frac{21,875}{216\eta^3} + \ldots + \delta_2\left\{2\eta + \frac{8}{3\eta} + \frac{1480}{81\eta^3} + \ldots\right\},$$

and hence, by multiplication with (105),

$$\tfrac{4}{5}\delta_0\xi_1(\delta_0\xi_2 + \tfrac{8}{5}\delta_2\xi_0) + (\delta_0\xi_2 + \tfrac{8}{5}\delta_2\xi_0)(3\xi_0^2 - \tfrac{10}{3})$$

$$= \tfrac{2025}{56}\eta^5 - \tfrac{127,125}{896}\eta^3 + \tfrac{3125}{14}\eta - \frac{78,125}{352\eta} + \ldots$$

$$+ \delta_2\left\{\tfrac{9}{2}\eta^3 + 6\eta + \frac{620}{9\eta} + \ldots\right\}.$$

We have from equations (89) and (100)

$$\xi_0 = \tfrac{1}{2}\eta + \frac{5}{3\eta} + \frac{50}{27\eta^3} + \frac{1000}{243\eta^5} + \ldots$$

$$\xi_1^2 = 9\eta^4 - 20\eta^2 + \tfrac{100}{9} + \frac{2000}{9\eta^2} + \ldots \quad ,$$

and hence, by multiplication, we find

$$\xi_0\xi_1^2 = \tfrac{9}{2}\eta^5 + 5\eta^3 - \tfrac{100}{9}\eta + \frac{3500}{81\eta} + \ldots \quad .$$

Therefore $(\delta_0\xi_1)^2(3\xi_0) = \tfrac{675}{128}\eta^5 + \tfrac{375}{64}\eta^3 - \tfrac{625}{48}\eta + \dfrac{21,875}{144\eta} + \ldots$.

From equations (91) and (89) we have

$$\tfrac{180}{7}\xi_0^3 - \tfrac{2825}{28}\xi_0 = \tfrac{45}{14}\eta^3 - \tfrac{1025}{56}\eta - \frac{2125}{84\eta} + \frac{154,375}{784\eta^3} + \ldots \quad ,$$

and from equation (93),

$$\delta_0\xi_1 = \tfrac{15}{8}\eta^2 - \tfrac{25}{12} + \frac{625}{27\eta^4} + \ldots$$

By multiplication of these last two series,

$$\delta_0\xi_1(\tfrac{180}{7}\xi_0^2 - \tfrac{2825}{28}\xi_0) = \tfrac{675}{112}\eta^5 - \tfrac{2625}{64}\eta^3 - \tfrac{3125}{336}\eta + \frac{6250}{7\eta} + \ldots \quad .$$

We have also

$$\delta_0\delta_2\tfrac{48}{25}\xi_0\xi_1 = \tfrac{6}{5}\delta_2\left\{\tfrac{1}{2}\eta + \frac{5}{3\eta} + \frac{50}{27\eta^3} + \ldots\right\}\{3\eta^2 - \tfrac{10}{3} + 0\,.\,\eta^{-2} + \ldots\}$$

$$= \delta_2\{\tfrac{9}{5}\eta^3 + 4\eta + 0\,.\,\eta^{-1} + \ldots\}.$$

The last series we require is ξ_0^5. By multiplication of the series (91) and (92), we find

$$\xi_0^5 = \tfrac{1}{32}\eta^5 + \tfrac{25}{48}\eta^3 + \tfrac{875}{216}\eta + \frac{5000}{243\eta} + \ldots \quad .$$

If we now collect the various series which have been obtained and substitute them in equation (102), we find, as the equivalent of this equation,

$$\left[\tfrac{3}{8}\eta + \frac{5}{6\eta} - \frac{25}{27\eta^3} - \frac{500}{243\eta^5} - \ldots\right]\left[5e_5\eta^4 + 3e_3\eta^2 + e_1 - \frac{e_{-1}}{\eta^2} - \ldots\right]$$
$$+ \delta_2[\tfrac{9}{5}\eta^3 - 6\eta + 0 \,.\, \eta^{-1} + \ldots]$$
$$= \delta_2\left[\tfrac{9}{5}\eta^3 + 6\eta + \frac{620}{9\eta}\right] + \delta_2[\tfrac{9}{5}\eta^3 + 4\eta + 0\,.\,\eta^{-1} + \ldots]$$
$$+ \tfrac{2025}{56}\eta^5 - \tfrac{127,125}{896}\eta^3 + \tfrac{3125}{14}\eta - \frac{78,125}{352\eta} + \ldots$$
$$+ \tfrac{675}{128}\eta^5 + \tfrac{375}{64}\eta^3 - \tfrac{625}{48}\eta + \frac{21,875}{144\eta} + \ldots$$
$$+ \tfrac{675}{112}\eta^5 - \tfrac{2625}{64}\eta^3 - \tfrac{3125}{336}\eta + \frac{6250}{7\eta} + \ldots$$
$$+ e_5\left(\tfrac{33}{32}\eta^5 + \tfrac{25}{48}\eta^3 + \tfrac{875}{216}\eta + \frac{5000}{243\eta}\right)$$
$$+ e_3\left(\tfrac{9}{8}\eta^3 + \tfrac{5}{4}\eta + \frac{50}{9\eta} + \ldots\right)$$
$$+ e_1\left(\tfrac{3}{2}\eta + \frac{5}{3\eta} + \ldots\right)$$

Equating the coefficients of the various powers of η, we obtain

$$\tfrac{15}{8}e_5 = \tfrac{6075}{128} + \tfrac{33}{32}e_5 \quad \ldots\ldots \quad (106),$$
$$-\tfrac{25}{6}e_5 + \tfrac{9}{8}e_3 - \tfrac{9}{2}\delta_2 = -\tfrac{158,625}{896} + \tfrac{25}{48}e_5 + \tfrac{9}{8}e_3 \quad \ldots \quad (107),$$
$$-\tfrac{125}{27}e_5 - \tfrac{5}{2}e_3 + \tfrac{3}{8}e_1 - 16\,\delta_2 = \tfrac{5625}{28} + \tfrac{875}{216}e_5 + \tfrac{5}{4}e_3 + \tfrac{3}{2}e_1 \quad \ldots \quad (108),$$
$$-\tfrac{2500}{243}e_5 - \tfrac{25}{9}e_3 - \tfrac{5}{6}e_1 - \tfrac{620}{9}\delta_2 - \tfrac{3}{8}e_{-1} = \tfrac{440,625}{1008} + \tfrac{5000}{243}e_5 + \tfrac{50}{9}e_3 + \tfrac{5}{3}e_1 \quad (109).$$

Solving the first two of these equations, we obtain

$$e_5 = \tfrac{225}{4} \quad \ldots \quad (110), \qquad \delta_2 = -\tfrac{8625}{448} \quad \ldots \quad (111).$$

Equations (108) and (109) may be written

$$-\tfrac{625}{72}e_5 - \tfrac{15}{4}(e_3 + \tfrac{3}{10}e_1) - 16\delta_2 = \tfrac{5625}{28} \quad \ldots\ldots \quad (112),$$
$$-\tfrac{2500}{81}e_5 - \tfrac{25}{3}(e_3 + \tfrac{3}{10}e_1) - \tfrac{620}{9}\delta_2 - \tfrac{3}{8}e_1 = \tfrac{440,625}{1008} \quad \ldots \quad (113).$$

Multiply (112) and (113) by 20 and 9 respectively, and subtract, and we obtain

$$\tfrac{625}{6}e_5 + 300\delta_2 - \tfrac{15}{2}e_{-1} = \tfrac{9375}{112} \quad \ldots\ldots \quad (114).$$

From equations (110) and (111) we obtain

$$\tfrac{625}{6}e_5 + 300\delta_2 = \tfrac{9375}{112},$$

and hence, by comparison with (114), $e_{-1} = 0$.

This vanishing of e_{-1} supplies a searching test of the accuracy of the work. Equation (112) becomes, after simplification,

$$e_3 + \tfrac{3}{10}e_1 = -\tfrac{17075}{168} \quad \ldots\ldots \quad (115).$$

Our equations do not enable us to determine e_3 and e_1 separately; they are, however, all satisfied by taking

$$e_3 = -\tfrac{17075}{168} + \lambda \quad \ldots \quad (116), \qquad e_1 = -\tfrac{10}{3}\lambda \quad \ldots \quad (117)$$

where λ is unknown. We therefore have (equation 103)

$$\xi_3 = \tfrac{1125}{4}\eta^4 - \tfrac{17075}{56}\eta^2 + \lambda\,(3\eta^2 - \tfrac{10}{3}) + \text{terms in } \eta^{-4},\ \eta^{-6}, \text{ \&c.} \quad . \quad (118).$$

Using the value of δ_2 given by equation (111), we find as the values of d_4, d_2, and d_0 (p. 91),

$$d_4 = \tfrac{45}{7}, \qquad d_2 = -\tfrac{965}{14}, \qquad d_0 = \tfrac{69875}{504}.$$

§ 24. Collecting the values of the various constants, we find as the equation to the surface (equation 70),

$$\begin{aligned}\xi\eta = 1 + \tfrac{2}{5}(\xi^2 + \eta^2) &+ \{(\xi^3 + \eta^3) - \tfrac{10}{3}(\xi + \eta)\}\,(\theta + \lambda\theta^3) \\ &+ \theta^2\,\{\tfrac{45}{7}(\xi^4 + \eta^4) - \tfrac{965}{14}(\xi^2 + \eta^2) + \tfrac{69875}{504}\} \\ &+ \theta^3\,\{\tfrac{225}{4}(\xi^5 + \eta^5) - \tfrac{17075}{168}(\xi^3 + \eta^3)\} + \text{terms in } \theta^4,\ \theta^5, \text{ \&c.} \quad . \quad (119).\end{aligned}$$

The occurrence of the indeterminate quantity λ can easily be accounted for. For if we have a solution

$$\xi\eta = a^2 + \theta f_1 + \theta^2 f_2 + \theta^3 f_3 + \ldots \quad . \quad (120),$$

corresponding to a parameter θ which is connected with the rotation by the relation

$$1 - \omega^2/2\pi\rho = \delta_0 + \delta_2\theta^2 + \delta_3\theta^3 + \ldots \quad . \quad (121),$$

then we can obtain this same solution in another form by replacing the parameter θ by a new parameter $\theta + \lambda\theta^3$. As far as θ^3 this leaves the relation (121) between ω^2 and θ unaltered, whilst the equation to the surfaces as far as θ^3 becomes

$$\xi\eta = a^2 + (\theta + \lambda\theta^3) f_1 + \theta^2 f_2 + \theta^3 f_3 + \ldots \quad . \quad (122).$$

It accordingly appears that in equation (119) the value of λ is entirely at our disposal. We shall therefore take $\lambda = 0$.

Investigation of Stability.

§ 25. There is a large *à priori* probability that the linear series we are now considering will be stable for small values of θ, but it will be well to rigorously examine the question.

It appears from POINCARÉ'S researches that the whole investigation reduces to determining whether the angular momentum is a maximum or a minimum at $\theta = 0$.* We therefore require to calculate the angular momentum as far as θ^2, and the answer to our problem will depend upon the sign of the term containing θ^2.

As far as θ^2, the equation to the surface in polar co-ordinates (r, ϕ) is (equation (119))

$$\begin{aligned}r^2 = 1 + \tfrac{4}{5}r^2\cos 2\phi &+ 2\theta\,(r^3\cos 3\phi - \tfrac{10}{3}r\cos\phi) \\ &+ \theta^2\,(\tfrac{90}{7}r^4\cos 4\phi - \tfrac{965}{7}r^2\cos 2\phi + \tfrac{69875}{504}) \quad . \quad (123).\end{aligned}$$

The moment of inertia is given by

$$\mathrm{I} = \iint r^2\,dr\,r\,d\phi$$

* H. POINCARÉ, "Sur la Stabilité de l'Equilibre des Figures Pyriformes . . . . .," 'Phil. Trans.,' A, vol. 198, p. 333.

taken over the surface, and therefore by

$$\mathrm{I} = \tfrac{1}{2}\int_0^\pi r^4\, d\phi \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (124),$$

where r is given by equation (123).

Let us assume a solution as far as θ^2 of the form

$$r^4 = \alpha^4 (1 + \beta\theta + \gamma\theta^2) \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (125),$$

where α, β, γ are functions of ϕ only, α being given by

$$\alpha^2 (1 - \tfrac{4}{5}\cos 2\phi) = 1. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (126).$$

The value of r^2 corresponding to solution (125) is

$$r^2 = \alpha^2 (1 + \tfrac{1}{2}\beta\theta + \theta^2 (\tfrac{1}{2}\gamma - \tfrac{1}{8}\beta^2)),$$

and if we substitute this value for r^2, and the similar values for r and r^3 in equation (123), and equate the coefficients of θ and θ^2, we obtain

$$\tfrac{1}{2}\beta = 2 (\alpha^3 \cos 3\phi - \tfrac{10}{3}\alpha \cos\phi) \quad . \quad . \quad . \quad . \quad . \quad . \quad (127),$$

$$\tfrac{1}{2}\gamma - \tfrac{1}{8}\beta^2 = \tfrac{3}{2}\alpha^3\beta \cos 3\phi - \tfrac{5}{3}\alpha\beta\cos\phi + \tfrac{90}{7}\alpha^4 \cos 4\phi - \tfrac{965}{7}\alpha^2 \cos 2\phi + \tfrac{69875}{504},$$

whence, by elimination of β,

$$\tfrac{1}{2}\gamma = 8 (\alpha^3 \cos 3\phi - \tfrac{10}{3}\alpha \cos\phi)(\alpha^3 \cos 3\phi - \tfrac{5}{3}\alpha\cos\phi)$$
$$+ \tfrac{90}{7}\alpha^4 \cos 4\phi - \tfrac{965}{7}\alpha^2 \cos 2\phi + \tfrac{69875}{504}.$$

We can eliminate ϕ from this equation by the help of equation (126). The resulting value for $\frac{1}{2}\gamma$ contains only even powers of α; if we simplify this, and transform the numerical coefficients to decimals, we find

$$\tfrac{1}{2}\gamma = 68{\cdot}1\alpha^6 - 319{\cdot}8\alpha^4 + 264{\cdot}4\alpha^2 + 159{\cdot}4.$$

Now we require to find the coefficient of θ^2 in I (equation (124)), and this is $\frac{1}{2}\int_0^\pi \alpha^4\gamma\, d\phi$. We therefore require to know the value of $\int_0^\pi \alpha^{2n}\, d\phi$ for $n = 2, 3, 4, 5$. This integral can easily be evaluated for all positive integral values of n; the values which we require at present are as follows:—

$n =$	2,	3,	4,	5,
$\dfrac{1}{\pi}\int_0^\pi \alpha^{2n}\, d\phi =$	4·6	17·0	70·1	305·3

Substituting these values, we find at once that $\frac{1}{2}\int_0^\pi \alpha^4\gamma\, d\phi$ is a positive quantity.

Thus the moment of inertia and the angular velocity both increase as θ^2 increases from the value $\theta = 0$. The moment of momentum is therefore a *minimum* for the value $\theta = 0$, and this proves the stability of the series of pear-shaped figures.

The Series of Pear-Shaped Curves.

§ 26. The equation to the curves of this linear series has already been calculated as far as θ^3. In order to obtain a still better idea of the shape of the curves I have carried the calculation two degrees further. The calculation of these last two degrees is extremely heavy, and I have omitted all details in order to save space. The method is precisely similar to that which was followed in the calculations of §§ 20–23.

It was found that the coefficients multiplying terms in θ^4 and θ^5 were inconveniently large, and to obviate this, the parameter has been changed from θ to $10^{3/2}\theta$. After making this change we find, as far as θ^5, for the equation to the surface expressed in polar co-ordinates, and for the equation determining ω^2,

$$\begin{aligned} r^2 = {} & (1 + {\cdot}139\theta^2 + {\cdot}023\theta^4 + \ldots) - {\cdot}211\theta r \cos\phi \\ & + ({\cdot}8 - {\cdot}138\theta^2 - {\cdot}069\theta^4 + \ldots)\, r^2 \cos 2\phi \\ & + ({\cdot}063\theta - {\cdot}0064\theta^3 - {\cdot}0031\theta^5 \ldots)\, r^3 \cos 3\phi \\ & + ({\cdot}013\theta^2 + {\cdot}0008\theta^4 + \ldots)\, r^4 \cos 4\phi \\ & + ({\cdot}0036\theta^3 + {\cdot}00093\theta^5 \ldots)\, r^5 \cos 5\phi \\ & + ({\cdot}0011\theta^4 + \ldots)\, r^6 \cos 6\phi \\ & + ({\cdot}00043\theta^5 + \ldots)\, r^7 \cos 7\phi + \&c. \qquad (128). \end{aligned}$$

$$1 - \omega^2/2\pi\rho = {\cdot}625 - {\cdot}019\theta^2 - {\cdot}016\theta^4. \qquad (129).$$

§ 27. We must next consider within what limits the calculated terms of equation (128) will give a good approximation to the complete equation. It is clear that for given values of r and θ the worst approximation may be expected when $\phi = 0$. Let us therefore consider the function $\Phi(r, \theta)$, defined by

$$\begin{aligned} \Phi(r, \theta) = {} & (1 + {\cdot}139\theta^4 + {\cdot}023\theta^4 + \ldots) - {\cdot}211\theta r \\ & - ({\cdot}2 + {\cdot}138\theta^2 + {\cdot}069\theta^4 + \ldots)\, r^2 \\ & + ({\cdot}063\theta - {\cdot}0064\theta^3 - {\cdot}0031\theta^5)\, r^3 \\ & + ({\cdot}013\theta^2 + {\cdot}0008\theta^4 + \ldots)\, r^4 + ({\cdot}0036\theta^3 + {\cdot}00093\theta^5)\, r^5 \\ & + ({\cdot}0011\theta^4 + \ldots)\, r^6 + ({\cdot}00043\theta^5 + \ldots)\, r^7 + \&c. \qquad (130). \end{aligned}$$

The value of $\Phi(r, \theta)$ is expressed by a doubly infinite series, of which only a few terms are known. When $r = 0$, $\theta = 0$, the value of Φ is known to be accurately equal to unity. For small values of r and θ, equation (130) will give Φ with considerable accuracy, but for larger values of r and θ, the terms calculated will be inadequate to give a good approximation to the value of Φ. What then, we inquire, are the values of r and θ over which this approximation may be regarded as good?

The coefficient of each power of r is an infinite series of powers of θ, of which all terms up to θ^5 have been calculated. A glance at these series will show that the approximation is tolerably good so long as $\theta^2 < 1$, but begins to break down as soon as θ exceeds this unit value.

Supposing that we have assigned to θ some definite value less than unity, the value of $\Phi(r, \theta)$ will be given by an infinite series of powers of r, of which only the first seven are known. For small values of r these first few terms will give a sufficiently good approximation, for larger values the approximation will be bad.

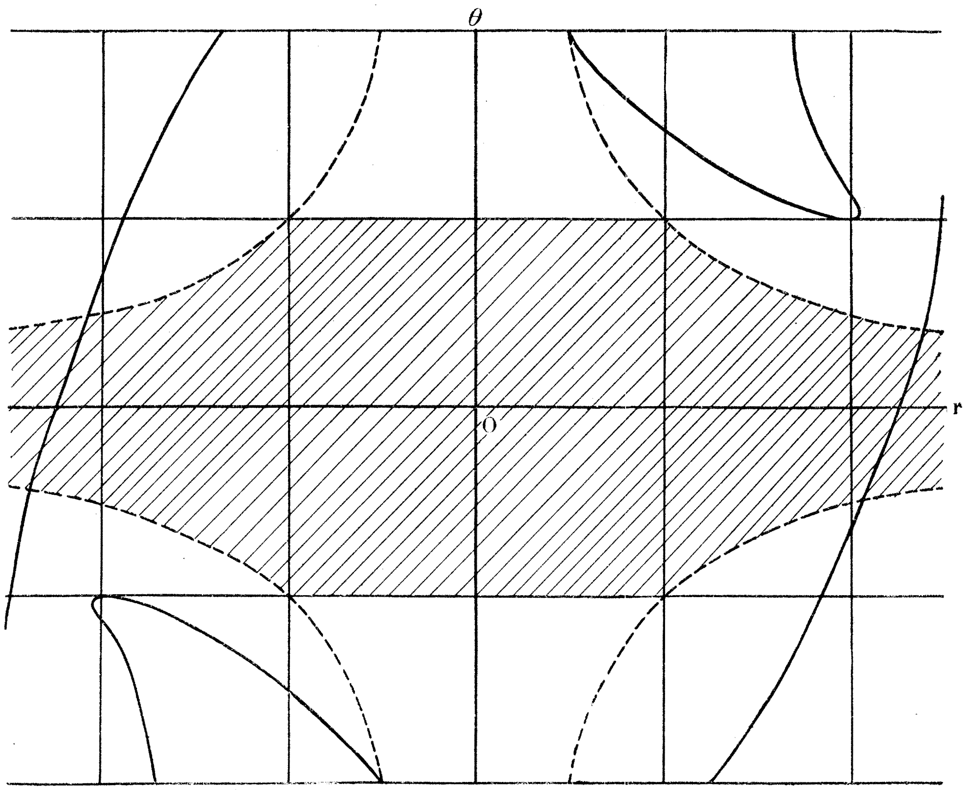

Fig. 3.

and for still greater values the series will become divergent, so that the first few terms give no approximation at all. It will be seen from inspection of equation (128) that the approximation will be tolerably good so long as $r^2 < 1/\theta^2$.

The conditions under which the calculated terms will give a good approximation may accordingly be supposed to be that $\theta^2 < 1$, and $r^2 < 1/\theta^2$. In fig. 3 is represented the plane of r, θ. The part of this plane over which the approximation is good is that bounded by the four curves

$$\theta = 1, \qquad \theta = -1, \qquad r\theta = 1, \qquad r\theta = -1.$$

This is the portion which is shaded in the figure.

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In this same figure the thick curves represent the locus,

$$\Phi(r, \theta) = 0,$$

calculated upon the supposition that the calculated terms of $\Phi(r, \theta)$ give a sufficiently good approximation to the whole. For the greater part of the curve this assumption is not justifiable, so that the curve requires adjustment, the amount of this adjustment increasing as we recede from the shaded portions of the plane. The most important points on the curve are those at which $d\theta/dr = 0$. These may with sufficient accuracy for our present purpose be taken to be $r = 2$, $\theta = 1$, and $r = -2$, $\theta = -1$.

§ 28. The points at which the axis $\phi = 0$ meets the curve of which the equation is (128) are given by

$$\Phi(r, \theta) = 0.$$

Fig. 3 accordingly enables us to trace the motion of these points as we move along the linear series, *i.e.*, as θ increases from zero upwards. At $\theta = 0$ we have, of course, two equal and opposite roots—$r = \pm\sqrt{5}$. As θ increases the positive root increases,

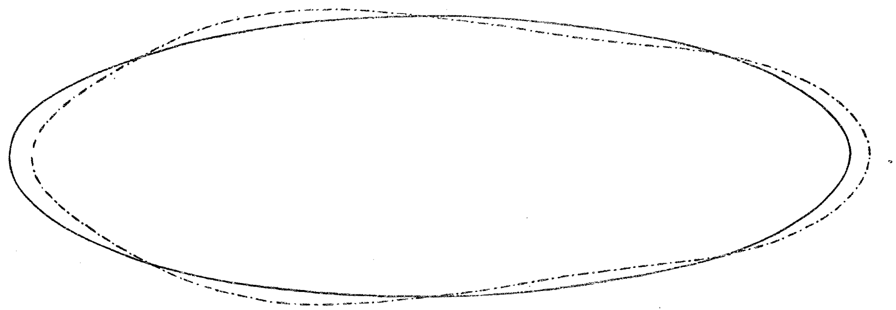

Fig. 4.

while the negative root numerically decreases. Remembering that the centre of gravity of the curve must remain at the origin, we see that this indicates a general thickening of the half of the curve in which $\phi > \pi/2$, with a diminution in the thickness of the forward half, and consequent lengthening of this half. These features become more marked as θ increases, until we reach the value $\theta = 1$, at which a new feature presents itself. For here there are two new roots occurring at the point $r = 2$. This indicates that the fluid separates into two portions when the value $\theta = 1$ is reached, the point of separation being $r = 2$ (approximately). We are at once struck by the great inequality in size between the primary and satellite: the former extending approximately from $r = -2$ to $r = +2$, and the latter only from $r = 2$ to $r = 2\frac{1}{2}$. The ratio of the linear dimensions will therefore be something like 8 to 1, but it must be remembered that our results require considerable correction on account of the imperfections in our approximations.

§ 29. In fig. 4 the thick curve is the elliptic cylinder corresponding to $\theta = 0$, and the dotted curve is the adjacent curve corresponding to a small value of θ ($\theta = \frac{1}{5}$).

In figs. 5 and 6 the curves are those corresponding to $\theta^2 = \frac{1}{2}$ and $\theta^2 = 1$. A glance at fig. 3 will show that there are difficulties in the way of drawing these latter curves with much accuracy. I have given in detail some of the calculations used in drawing the curve $\theta^2 = 1$, in order that the reader may judge for himself as to the closeness or otherwise of the approximations. The curve $\theta^2 = \frac{1}{2}$ is of course much easier. Before passing on to the calculations, two points ought to be noticed.

(i.) It will be noticed that in the various θ-series (the coefficients of powers of r in equation (128)) the terms last calculated are without exception of the same sign as those previously calculated. There is therefore some justification for hoping that the remainders in these series will be of the same sign as these last terms. If this is so, the error introduced by the neglect of these remainders could, to an appreciable extent, be reduced by an adjustment in the value of θ. Thus we shall be attempting to calculate the curve for (say) $\theta = 1$, and shall obtain a curve which is much more like the curve for some smaller value of θ (say $\theta = \cdot 98$) than it is like the curve $\theta = 1$. Regarded as an attempt at tracing a surface of equilibrium the error will be much smaller than if regarded as an attempt at tracing the particular curve $\theta = 1$.

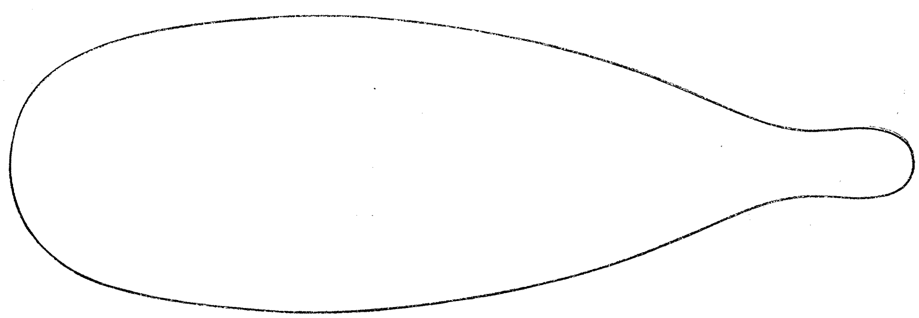

Fig. 5.

(ii.) It will be noticed that the sign of the leading terms in each of the series multiplying r^4, r^5, r^6, r^7, is positive. An examination of the method by which these leading terms are calculated will show that this is a general law: all the leading terms after r^3 are of positive sign. Thus the error will be reduced by supposing the series (128) continued to higher powers of r by a suitably chosen series of terms. I have accordingly done this in the calculations, and the conjectural terms are, throughout, enclosed in square brackets.*

The Curve $\theta^2 = \frac{1}{2}$. (Fig. 5.)

§ 30. In tracing this curve I started from $\phi = \pi$, and calculated a series of points on the curve for decreasing values of ϕ. The approximation at $\phi = \pi$ was found to

* The effect of these corrections must, of course, be small; but, at the same time, it seems as well to make use of any definite knowledge that we possess.

be good, the root being 1·98, and the error occurring only in the third decimal place. As ϕ decreases the approximation improves, and at $\phi = \pi/2$ the error occurs only in the fifth decimal place. At $\phi = 7\frac{1}{2}°$ the error again appears in the third place, and after this the approximation is bad. My plan was to calculate for smaller values of ϕ as well as I could, taking care to keep the values of r in defect rather than excess of their true values. The curve was then plotted out on paper ruled with squares of 1 millim., the unit of length being taken to be 50 millims.*

The area of the elliptic cylinder of fig. 4 is known to be

13,090 sq. millims.,

and this would also have been the area of the present curve had it been accurately drawn. The area of the curve (obtained by counting squares) was, however, found to be

12,776 sq. millims.

The moments about the axis $\phi = \pi/2$ of the two parts of the curve ($\phi > \pi/2$ and

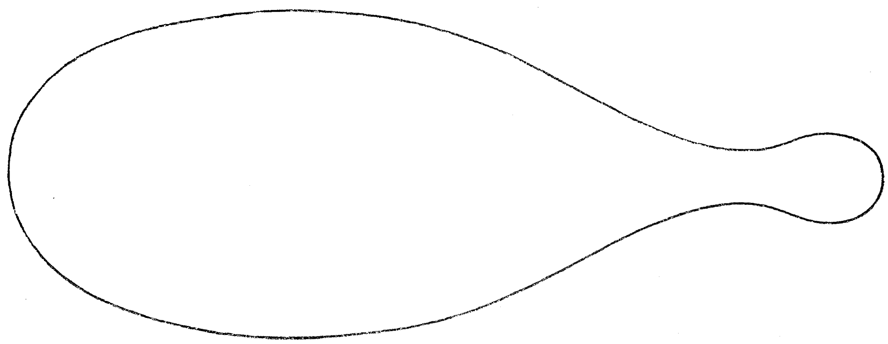

Fig. 6.

$\phi < \pi/2$) ought of course to be equal: these were found to be respectively

298,290 cub. millims. and 302,850 cub. millims.

It was therefore obvious that the curve had been too much shortened in the region in which $\phi < 7\frac{1}{2}°$.

Readjusting the curve in this region so as to divide the error as equally as possible, I arrived at a curve with outstanding errors in area and moment of

139 sq. millims. and $-$ 139 sq. millims. at $r = 125$ millims., $\phi = 0$.

This is the curve given in fig. 5. It will be seen that the error is one of about 1 per cent.

Calculation of the Curve $\theta = 1$. (Fig. 6.)

§ 31. The equation of the curve is found to be

$$r^2 = 1{\cdot}162 - {\cdot}211 r \cos\phi + {\cdot}593 r^2 \cos 2\phi + {\cdot}053 r^3 \cos 3\phi + {\cdot}014 r^4 \cos 4\phi + {\cdot}0045 r^5 \cos 5\phi + {\cdot}0011 r^6 \cos 6\phi + {\cdot}00043 r^7 \cos 7\phi + [{\cdot}0002 r^8 \cos 8\phi + {\cdot}0001 r^9 \cos 9\phi + \ldots]$$

the terms in square brackets being those mentioned at the end of § 29.

* This has been photographically reduced to 25 millims. before printing.

We calculate in succession the value of the radius vector when $\phi = 180°$, $150°$, $120°$, $90°$, $60°$, $45°$, &c. The value of r is a root of $\Phi(r) = 0$, where

$$\Phi(r) = -r^2 + 1{\cdot}162 - {\cdot}211 r \cos\phi + \&c.$$

Our method will be to find a value of r for which $\Phi(r)$ is small, and calculate $\Phi(r)$ and $d\Phi/dr$ at this point. The true root R is then given by

$$R = r - \frac{\Phi(r)}{d\Phi(r)/dr}.$$

When $\phi = \pi$, $\Phi = 1{\cdot}162 + {\cdot}211r - {\cdot}407r^2 - {\cdot}053r^3 + {\cdot}014r^4 - {\cdot}0045r^5 + {\cdot}0011r^6 - {\cdot}00043r^7 + [{\cdot}0002r^8 - {\cdot}0001r^9 + \ldots]$.

When $r = 1{\cdot}8$, $\Phi = 1{\cdot}162 + {\cdot}380 - 1{\cdot}319 - {\cdot}310 + {\cdot}147 - {\cdot}085 + {\cdot}038 - {\cdot}026 + [{\cdot}020 - {\cdot}018 + \ldots] = -{\cdot}003 + [{\cdot}01]$,

$$d\Phi/dr = {\cdot}21 - 1{\cdot}4 - {\cdot}5 + {\cdot}3 - {\cdot}2 + {\cdot}1 - \ldots = -1{\cdot}4,$$

therefore $R = 1{\cdot}8 - {\cdot}002 + [{\cdot}007]$, or, $R = 1{\cdot}805$.

When $\phi = 5\pi/6$, $\Phi = 1{\cdot}162 + {\cdot}182r - {\cdot}704r^2 - {\cdot}007r^4 + {\cdot}0039r^5 - {\cdot}0011r^6 + {\cdot}00036r^7 - [{\cdot}0001r^8 + {\cdot}0000r^{10} + \ldots]$.

When $r = \sqrt{2} = 1{\cdot}414$, $\Phi = 1{\cdot}162 + {\cdot}257 - 1{\cdot}408 - {\cdot}028 + {\cdot}022 - {\cdot}009 + {\cdot}004 - [{\cdot}001] = {\cdot}000 - [{\cdot}001]$,

therefore $R = 1{\cdot}414$.

When $\phi = \pi/2$, $\Phi = 1{\cdot}162 - 1{\cdot}593r^2 + {\cdot}014r^4 - {\cdot}0011r^6 + [{\cdot}0002r^8] + \ldots$

When $r^2 = {\cdot}74$, $r = {\cdot}859$;

$$\Phi = 1{\cdot}162 - 1{\cdot}179 + {\cdot}007 - {\cdot}0003 + [{\cdot}0000] + \ldots = -{\cdot}010;$$

$d\Phi/dr = -2{\cdot}5$; therefore $R = {\cdot}859 - {\cdot}004 = {\cdot}855$.

When $\phi = 7\tfrac{1}{2}°$, $\Phi = 1{\cdot}162 - {\cdot}209r - {\cdot}427r^2 + {\cdot}046r^3 + {\cdot}012r^4 + {\cdot}0036r^5 + {\cdot}0008r^6 + {\cdot}0003r^7 + [{\cdot}0001r^8 + {\cdot}00004r^9 + \ldots]$.

When $r^2 = 3$, $r = 1{\cdot}732$, $\Phi = 1{\cdot}162 - {\cdot}348 - 1{\cdot}281 + {\cdot}241 + {\cdot}108 + {\cdot}056 + {\cdot}021 + {\cdot}014 + [{\cdot}008 + {\cdot}005 + \ldots] = -{\cdot}027 + [{\cdot}01]$.

$d\Phi/dr = -{\cdot}209 - 1{\cdot}44 + {\cdot}42 + {\cdot}23 + {\cdot}17 + {\cdot}08 + \ldots = -{\cdot}75$; therefore $R = 1{\cdot}732 - {\cdot}036 + [{\cdot}01] = 1{\cdot}71$.

When $\phi = 4°$, $\Phi = 1{\cdot}162 - {\cdot}211r - {\cdot}410r^2 + {\cdot}052r^3 + {\cdot}014r^4 + {\cdot}004r^5 + {\cdot}001r^6 + {\cdot}0004r^7 + [{\cdot}00017r^8 + {\cdot}00008r^9 + \ldots]$.

When $r = 2$, $\Phi = 1{\cdot}162 - {\cdot}422 - 1{\cdot}640 + {\cdot}416 + {\cdot}224 + {\cdot}128 + {\cdot}064 + {\cdot}051 + [{\cdot}05 + {\cdot}04 + \ldots] = -{\cdot}02 + [{\cdot}1\ ?]$.

$d\Phi/dr = -{\cdot}211 - 1{\cdot}640 + {\cdot}624 + {\cdot}448 + {\cdot}320 + {\cdot}192 + {\cdot}178 + [{\cdot}18 + {\cdot}18 + \ldots] = {\cdot}01 + [\ ?\ ]$.

When $r = 1{\cdot}8$, $\Phi = 1{\cdot}162 - {\cdot}390 - 1{\cdot}328 + {\cdot}297 + {\cdot}147 + {\cdot}076 + {\cdot}034 + {\cdot}023 + [{\cdot}017 + {\cdot}014 + \ldots] = {\cdot}02 + [{\cdot}05]$.

$d\Phi/dr = -{\cdot}211 - 1{\cdot}47 + {\cdot}49 + {\cdot}33 + {\cdot}21 + {\cdot}11 + {\cdot}09 + [{\cdot}08 + {\cdot}07 + \ldots] = -{\cdot}45 + [{\cdot}2]$.

There is therefore a root at about $r = 1 \cdot 95$, but it is clear that we are already in the region at which the approximation ceases to be satisfactory, and that we are close upon the region in which the series become actually divergent.

The smallness of $d\Phi/dr$ indicates that somewhere in the neighbourhood of the point just calculated we come to a point at which there is a pair of equal roots for r, and therefore a "minimum" in the value of ϕ. This is the "neck" of which the first signs are apparent in fig. 5. Let us refer to all the matter to the left of this "neck" as the "primary," to all that to the right as the "satellite." Let the exact line of division be a vertical line at a distance 2 from the origin.

The primary has been drawn with fair accuracy; the satellite must be drawn in the manner adopted in the difficult region of the former curve.

The area of the primary was found to be

11778 sq. millims.,

and this leaves 1312 sq. millims. to be accounted for by the satellite and the error in drawing. The centre of gravity of the primary was found to be 4·7 millims. to the left of the origin. Distributing the error as equally as possible, I have arrived at the curve of fig. 6. The area of primary and satellite are respectively

11778 and 881 sq. millims.,

the error in area is a defect of 431 sq. millims., and that in the moment about $\phi = \pi/2$ is that of an excess of 431 sq. millims. at the point $r = 125$ millims., $\phi = 0$. The error in the whole curve is therefore about 3 per cent.; that in the satellite is unfortunately of the same order of magnitude as the satellite itself.

§ 32. The following table sums up the results which have been obtained, and also contains some new results. The moments of momentum of the last curves were obtained by a process of counting on squared paper, and are not carried to any great accuracy.

	Curve.		Area.	$\frac{\omega^2}{2\pi\rho}$.	Angular momentum $\left(\text{omitting factor } \sqrt{\frac{2\rho^3}{\pi}}\right)$.
(1)	Circle	—	1	0	0
(2)	,, ...	—	1	·43	·33
(3)	,, ...	Point of bifurcation . .	1	·5	·35
(4)	Ellipse	Ratio of axes $\sqrt{5} : 1$ . .	1	·43	·44
(5)	,, fig. 4 . .	Point of bifurcation: ratio of axes 3 : 1	1	·375	·51
(6)	Fig. 5	$\theta^2 = \frac{1}{2}$	1	·39	·53
(7)	Fig. 6	$\theta^2 = 1$	1	·42 ?	·55 ?
(8)	,, ...	Separation of fluid into primary and satellite	1	·43 ?	·57 ?
(9)	The two parts of curve (8)	Primary	·93 ?	·43 ?	·40 ?
(10)	The two parts of curve (8)	Satellite	·07 ??	·43 ?	·17 ??

In this table the quantities of which the numerical values are doubtful are marked with a query. A single query indicates a probable error of 1 or 2 per cent.; a double query indicates that the error may be comparable with the quantity itself.

Let us examine the state of things just after separation has taken place. The satellite is describing an orbit about the primary, both bodies rotating with the same angular velocity. This angular velocity is, to within a few per cent., given by

$$\omega^2/2\pi\rho = \cdot 43 \quad \text{. . . . .} \quad (131).$$

If the satellite exerted no attraction upon the primary, the figure of the primary would be a figure of equilibrium under the influence of a rotation given by (131). The force exerted by the satellite may be divided into two parts, a uniform force in the direction $\theta = 0$, and a tide-generating force of the usual kind. If the former of these existed alone, the configuration of the primary would still be one of equilibrium under a rotation of amount given by (131). We therefore see that the actual configuration of the primary may be regarded as a configuration of equilibrium under rotation given by (131), disturbed by the tide-generating potential which is caused by the satellite.

Since this tide-generating potential is small, except in the immediate neighbourhood of the satellite, it ought to be possible to remove the tides from the surface of the primary, and form a pretty good idea of the configuration which would be the configuration of the primary except for tidal disturbance. If this is done with the primary of fig. 6, it will be found that the remaining curve is a very good ellipse. We may therefore conjecture that curve (9) is an ellipse deformed by the tidal influence of its satellite.

Now the ellipse corresponding to the amount of rotation given by (131) is curve 4 of the preceding table. We see that the axes are in the ratio $\sqrt{5} : 1$, and this is in good agreement with the ellipse obtained by removing the tides in fig. 6. The momentum of the ellipse of unit area of which the axes are in the ratio $\sqrt{5} : 1$ (curve 4) is ·44. If we reduce this so as to apply to an ellipse of area ·93 instead of to one of unit area, we find an angular momentum of ·38. Since this ellipse must be supposed to rotate not about its centre, but about the centre of gravity of itself and a satellite about one-fifteenth of its mass, situated at the end of its axis, this angular momentum must be increased to about ·39. The small discrepancy between this and the value ·40 obtained for curve 9 may be accounted for partly by errors of approximation, and partly by the increase of momentum caused by the tidal deformation of the ellipse.

We can check our result in another way. The equation of the ellipse being

$$ax^2 + by^2 = 1 \quad \text{. . . . .} \quad (132),$$

the force at x, 0, a point near the extremity of the major axis and outside the ellipse, may, to a good approximation, be taken to be $\frac{2\omega\rho}{\sqrt{a}+\sqrt{b}}\frac{2}{\sqrt{a}\,.\,x}$, and if x, 0 is the centre of gravity of the satellite, this must be equal to

$$\cdot 93\omega^2 x,$$

the factor ·93 being introduced to allow for the displacement of the centre of gravity of the primary.

Taking $\sqrt{b} = \sqrt{(5a)}$, we find the equation

$$\frac{2}{1+\sqrt{5}}\frac{1}{ax^2} = \cdot 93\,\frac{\omega^2}{2\pi\rho},$$

and putting $\omega^2/2\pi\rho = \cdot 43$, this gives the values

$$ax^2 = 1\cdot 54, \qquad \sqrt{a}\,.\,x = 1\cdot 24.$$

Now $\sqrt{a}\,.\,x$ is the sum of the semi-axes of primary and satellite divided by the semi-axes of the primary. The equation just found is therefore about as true as could be expected, the linear diameters of primary and satellite being approximately in the ratio of 4 to 1.

The ellipse given as curve 4 is stable, and, since the mass of the satellite is small compared with that of the primary, we may suppose the combination of primary and satellite to be stable. Thus, if our conjecture as to the interpretation of curve 8 is sound, it appears that the linear series which commences with curve 5 remains stable until the mass separates into two masses.

The motion of a gradually-cooling mass will therefore be through the following cycle of changes. Firstly, increase of the ratio [angular momentum ÷ (area)^2] until we reach curve 5. Then motion along POINCARÉ's linear series until we reach curve 9. At this point separation takes place, and the primary is left as a tidally-distorted form of curve 4. As the satellite recedes the tidal distortion decreases, and as the value of the ratio [angular momentum ÷ (area)^2] again increases, the configuration moves along the Jacobian series of elliptic cylinders until curve 5 is again reached. This completes the cycle, and the continual repetition of the cycle can only be ended by solidification, or some similar cause which is outside our present considerations.

I have not attempted to give any discussion of results from the point of view of dynamical astronomy. The complications introduced by the heterogeneity and compressibility of natural substances, as well as by the difference between the two-dimensional and three-dimensional problems, are so great that any discussion with reference to the actual conditions of astronomy would be impossible in the present paper.

I have had the advantage of frequent conversations with Professor DARWIN on the subject of this paper; my thanks are also due to Professor FORSYTH for advice in connection with the earlier sections.